THE

PHYSICAL SOCIETY

OF

LONDON.

PROCEEDINGS.

VOLUME XXX.—PART I.

DECEMBER 15, 1917

Price to Non-Fellows, **4s.** *net, post free* **4/3.**

...ion, **20/-** *post free, payable in advance*

Published Bi-Monthly from December to August.

LONDON:
FLEETWAY PRESS, LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET.

1917.

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I. *The Radius of the Electron, and the Nuclear Structure of Atoms.* By Prof. J. W. Nicholson, M.A., D.Sc., F.R.S.

Received Aug. 22, 1917.

The present note is intended rather to make a suggestion than to formulate any definite theory of the structure of the nucleus of an atom according to the model at present found necessary in order to interpret such phenomena as radioactivity, atomic number, and scattering of charged particles by atoms. The electron is usually regarded as a kind of globule of electricity with a definite radius, and as the nuclei of the more complex atoms must, from certain considerations, be supposed to contain electrons, and at the same time preserve their minute size, a difficulty is encountered unless we may suppose that electrons and positive charges can actually in some way inter-penetrate each other and occupy the same space. Some means of removing the finite radius of an electron, and with it all discontinuity at a prescribed surface, is manifestly desirable. On theories such as that of Lorentz, the electron, a sphere when at rest, is deformed when in motion, but in all cases in which hypotheses as to the inner structure and internal equilibrium of an electron are introduced, it has been given, when at rest, this definite "radius," marking off a distinct region from the aether, and involving a discontinuity of some form at the boundary. However great the departure of this view from the more natural intuitions or prejudices of those who regard the electron as a structure built out of aether, it has been of great service at many points. Especially, in the hands of Lorentz and others, it has led to a conception of the variation of the mass of an electron with its speed, which is in undoubted agreement with careful experiments, and must contain a large substratum of truth. Such considerations involve the existence of a linear constant which is the same for every electron and is usually regarded as a "radius." A similar constant is, of course, necessary for the elementary positive charge.

If the terminology which makes use of an aether, out of which the elementary charges are constructed as regions of strain, is adopted, it would seem more natural that such line-constants should be constants with some significance through-

out the whole aether, rather than constants which only come into being when the aether is strained into the form of matter. The aether may, in fact, be in some manner cellular, with these linear magnitudes involved in the specification of the cells, and thereby in any strained structure composed from them. In this Paper we accordingly make a tentative suggestion towards this form of interpretation of the line-constants by regarding the electron as a state of strain which is for practical purposes concentrated at its "centre," rapidly diminishing outwards from this point according to some very convergent law, which involves a line-constant in its specification. It is found that no important difference is made in the mutual reactions of electrons except at distances comparable with their "radii," and that if the strains are regarded as capable of superposition, inter-penetration is readily possible without the introduction of indefinitely large forces between the components. A form in which neutral doublets could exist as a part of nuclear structure also becomes evident.

The mathematical treatment, on the basis of a simple exponential law of attenuation of the strain, is only an illustration developed for purposes of clearness. If the suggestion were to correspond in any way with reality, no phenomena at present available could give a clue to the actual law. An elementary argument on the basis of physical dimensions, however, is sufficient to show that any other law would only lead to certain differences in numerical coefficients.

Sir Joseph Larmor alone appears not to be definitely bound to the point of view of the finite electron. For the purposes of his theory,* an electron is a type of singularity, made up of aether in a peculiar state of strain, which needs no more precise mathematical definition of the state of strain than is implied in the relation

$$\iint (lf+mg+nh)ds=e,$$

or, the surface integral of aethereal polarisation over any surface surrounding one electron is $4\pi e$. This serves as a definition of e. In another form, an electron is a region in which the divergence of the aethereal polarisation is not zero. From this point of view, an electron might have no boundary, and the singularity could be confined effectively to a very small

* "Æther and Matter," Camb. Univ. Press, 7900, pp. 86–90.

region by choosing a distribution of electrical density ρ, defined by

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=\rho,$$

following any rapidly convergent law of variation, for example, $\rho=\varepsilon^{-\lambda r}$, where r is distance from a point. The departure of the surface integral from $4\pi e$ would be inappreciable within a few diameters of the electron, if a diameter is defined as a length comparable with λ^{-1}.

It is noteworthy that Lorentz, whose electron, of those investigated in detail, alone seems to give a reasonable description of those physical phenomena for which the nature of the electron is important, should have found it necessary to impose a contraction on the moving electron of like magnitude with the contraction of all the "interspaces" between electrons, found earlier to the second order by Larmor, by an analysis which is equally applicable to all orders. This equality of the contraction, whether regarded from the point of view of the Principle of Relativity or not, seems in itself to imply a continuity of the electron with the rest of space—an aethereal structure for the electron with no defined boundary. A subsequent brief survey of the consequences of the exponential formula already suggested, as a mere illustrative case, will indicate the bearing of this remark.

But a length is necessarily associated with an electron, as is even apparent from the existence of its electromagnetic mass, with its necessary dimensions. Such a length in the exponential case would be given by λ^{-1}. Whatever the physical significance of this length, it is evident that a view which attaches to it the same significance for all parts of an infinite aether is more acceptable on some grounds than one which relates it to a definite region only round the "centre" of the electron. The endowment of the whole aether with such a linear constant is ultimately equivalent to an endowment with some form of structure. Such a possibility of aethereal structure might ultimately solve many difficulties. It appears to be demanded by the recent quantum or unit theory of energy—provided that the view of an aether is retained—and it would involve an equality of λ for all electrons, since λ would no longer be a property of the electron, but of the whole aether. There is evidence that e, the charge of an electron, bears a relation to Planck's quantum constant h. These brief indications will

suffice to show that we can find a means of removing the boundary of an electron while retaining all the analysis of Lorentz, so that a reconciliation of the quantum theory with the necessary properties of the electron, and, in fact, a deduction of the one from the other, is a possible hope for the future. The analysis of the bounded electron, which becomes spheroidal when in motion, then appears as a convenient mathematical substitute for the more complete analysis of an electron of infinite extent, consisting of a state of strain impressed on a structural aether, but a strain of such a mathematical form that it is confined for physical purposes in a certain region of minute extent, comparable in diameter with a linear magnitude involved in the structure of the aethereal cells.

When, in the ordinary view, the radius of an electron is a, the usual formula for the mass of the electron, of electromagnetic origin, is

$$m_0=\frac{2}{3}\frac{e^2}{ac^2}$$

for speeds small in comparison with c, the velocity of light in free aether. In the present note, we shall not be concerned with a higher approximation on account of the comparatively small velocities which it is necessary to ascribe to the electrons in the model atoms which are found capable of giving some account of such phenomena as spectral series.

We proceed to a brief survey of the illustrative case of a more continuous electron, already indicated in the preceding pages. It is not claimed that this is more than a mere illustration of the possibilities of such a theory.

Let us define an electron-centre as an origin around which there is a symmetrical distribution of polarisation satisfying

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}=\rho=A\varepsilon^{-\lambda r},$$

where λ is a constant, the reciprocal of a certain length fixed by the aethereal structure. In polar co-ordinates—

$$\frac{1}{r^2}\frac{\partial}{\partial r}(fr^2)+\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}(g\sin\theta)+\frac{1}{r}\sin\theta\frac{\partial h}{\partial\varphi}=A\varepsilon^{-\lambda r},$$

where (fgh) are now radial and transversal components. In case of a dependence only on r,

$$\frac{\partial}{\partial r}(fr^2)=Ar^2\varepsilon^{-\lambda r},$$

and we may write $g=h=o$.

Thus $$fr^2 = -A\left\{\frac{r^2}{\lambda}+\frac{2r}{\lambda^2}+\frac{2}{\lambda^3}\right\}\varepsilon^{-\lambda r}+\text{constant}.$$

With the constant equal to $2A/\lambda^3$, f is finite at the origin, and in fact zero, so that

$$f=-A\left\{\frac{1}{\lambda}+\frac{2}{r\lambda^2}+\frac{2}{r^2\lambda^3}\right\}\varepsilon^{-\lambda r}+\frac{2A}{\lambda^3 r^2},$$

and at a great distance, the intensity of electric force follows the law of inverse square.

The total charge of the electron being e,

$$e=\iiint \rho\, dx\, dy\, dz$$

taken throughout space, or

$$e=A\int_0^\infty\int_0^{2\pi}\int_0^\pi \varepsilon^{-\lambda r}\, r^2 \sin\theta\, dr\, d\varphi\, d\theta$$

$$=4\pi A\int_0^\infty r^2\varepsilon^{-\lambda r}dr,$$

$$=8\pi A/\lambda^3,$$

so that at a sufficient distance

$$f=\frac{e}{4\pi r^2},$$

and the electric force is e/r^2. The true polarisation would be

$$f=\frac{e}{4\pi r^2}-\frac{e}{8\pi}\left\{\frac{2}{r^2}+\frac{2\lambda}{r}+\lambda^2\right\}\varepsilon^{-\lambda r}.$$

The electrostatic energy in the field becomes

$$W=\frac{1}{8\pi}\iiint (4\pi f)^2 r^2 d\omega\, dr,$$

or, after considerable reduction, at various stages

$$W=\frac{1}{2}e^2\int_0^\infty dr\left\{\frac{1}{r}-\left(\frac{1}{r}+\lambda+\frac{\lambda^2 r}{2}\right)\varepsilon^{-\lambda r}\right\}^2,$$

$$=\frac{\lambda e^2}{32}-\frac{\lambda e^2}{8}+\frac{1}{2}e^2\int_0^\infty \frac{dr}{r^2}\left\{1-(1+\lambda r)\varepsilon^{-\lambda r}\right\}^2,$$

$$=-\frac{3}{32}\lambda e^2+\lambda^2 e^2\int_0^\infty dr\,\varepsilon^{-\lambda r}\left\{1-(1+\lambda r)\varepsilon^{-\lambda r}\right\}$$

(integrating by parts)

$$=-\frac{3}{32}\lambda e^2+\frac{\lambda e^2}{4}=\frac{5}{32}\lambda e^2.$$

A more difficult problem is the determination of the mutual energy of two electrons occupying the aether at the same time, so that their fields are superposed. From the solution, a knowledge of a change in the law of force between the electrons at distances comparable with λ, can be obtained.

If for convenience we write

$$rf(r)=\frac{1}{r}-\left(\frac{1}{r}+\lambda+\frac{\lambda^2 r}{2}\right)\varepsilon^{-\lambda r},$$

and (r, r') are distances measured from the electrons, which are in the z axis, the portion of the field energy which is mutual is—

$$W_1=\frac{e^2}{4\pi}\iiint f(r)f(r')d\tau,$$

where $d\tau$ is an element of volume, or

$$W_1=e^2\int_0^\infty\int_0^\pi f(r)f(r')r^2\sin\theta drd\theta,$$

where $$r'=\sqrt{r^2+z^2-2rz\cos\theta},$$

z being the distance between the electron-centres.

We can proceed otherwise as follows: The electric forces (PQR), $(P'Q'R')$ of the electrons are derived from potentials V, V', and the mutual energy in any region is

$$\frac{1}{4\pi}\iiint\left(\frac{\partial V}{\partial x}\frac{\partial V'}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial V'}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial V'}{\partial z}\right)d\tau$$

$$=-\frac{1}{4\pi}\iint V'\frac{\partial V}{\partial n}dS-\frac{1}{4\pi}\iiint V'\nabla^2 V d\tau,$$

by the usual notation of Green's theorem, and the surface integral is negligible over the infinite boundary of the whole field, where V, V' are each of order $1/r$. Moreover, $\nabla^2 V' =4\pi\rho'$, and in the usual way,

$$W_1=\iiint V\rho' d\tau,$$

where the integral is taken over all space. This becomes

$$W_1=\frac{e\lambda^3}{8\pi}\iiint V\varepsilon^{-\lambda r'}r^2 drd\omega$$

$$=\tfrac{1}{4}e\lambda^3\int_0^\infty\int_0^\pi Vr^2\sin\theta\varepsilon^{-\lambda r'}drd\theta,$$

the origin being at the first electron, and $d\omega$ an element of solid angle. In order to obtain V, we note that it vanishes at

infinity, and that $-\partial V/\partial r$ is the electric intensity due to an electron situated at the origin. Thus

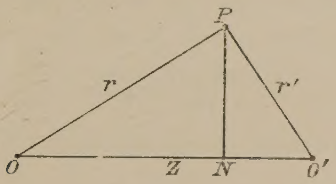
$$\frac{-\partial V}{\partial r}=\frac{e}{r^2}-e\left(\frac{1}{r^2}+\frac{\lambda}{r}+\frac{\lambda^2}{2}\right)\varepsilon^{-\lambda r}$$

$$V=\int_r^\infty\frac{edr}{r^2}-\int_r^\infty e\left(\frac{1}{r^2}+\frac{\lambda}{r}+\frac{\lambda^2}{2}\right)\varepsilon^{-\lambda r}dr$$

$$=\frac{e}{r}-\frac{e}{r}\left(1+\frac{\lambda r}{2}\right)\varepsilon^{-\lambda r}.$$

The potential at the centre of the electron is finite and equal to $\frac{1}{2}e\lambda$.

It is more convenient to use bipolar co-ordinates, for we must avoid, in the integrations, attaching a negative sign to r or r'.



Taking the centres O, O' of the electrons as origins, then any point, P, or (r, r') can equally be defined by the values of $r\pm r'$. Moreover, the curves $r\pm r'=$constant are the set of confocal ellipses and hyperbolas in any plane through O, O', with these points as foci. If $c=\frac{1}{2}z$, and $(\xi\eta)$ are the elliptic co-ordinates defined by

$$x=c\cosh\xi\cos\eta,\qquad y=c\sinh\xi\sin\eta,$$

then we readily find

$$(r, r')=c(\cosh\xi\pm\cos\eta),$$

which are essentially positive, since the minimum value of $\cosh\xi$ is unity. The semi-major axis of the ellipse through any point is, of course, $c\cosh\xi$, or $\dfrac{r+r'}{2}$.

Write $r+r'=\rho$, $r-r'=\sigma$, when r is greater than r'. Then ρ and σ are orthogonal co-ordinates in space, and their directions of increase are those of (ξ, η), or normal to the ellipse and hyperbola through any point in the plane of O, O'. The ele-

ments ds_1 and ds_2 of length along ρ and σ increasing are then given by

$$ds_1 = d\xi\left\{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2\right\}^{\frac{1}{2}} = c\sqrt{\cosh^2\xi - \cos^2\eta}\, d\xi$$

$$ds_2 = d\eta\left\{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2\right\}^{\frac{1}{2}} = c\sqrt{\cosh^2\xi - \cos^2\eta}\, d\eta$$

or
$$ds_1 ds_2 = c^2 d\xi d\eta(\cosh^2\xi - \cos^2\eta).$$

The third element may be defined by a rotation $d\varphi$ of the line PN round the axis O, O', and has a length

$$ds_3 = PNd\varphi = c\sinh\xi\sin\eta d\varphi.$$

Thus the volume of a rectangular element of space at the point P is

$$d\tau = c^3 d\xi d\eta d\varphi \sinh\xi\sin\eta(\cosh^2\xi - \cos^2\eta).$$

But
$$2c\cosh\xi = r + r' = \rho,\quad 2c\sinh\xi d\xi = d\rho$$
$$2c\cos\eta = r - r' = \sigma,\quad 2c\sin\eta d\eta = -d\sigma,$$

so that
$$d\tau = \frac{\rho^2 - \sigma^2}{16c} d\varphi d\rho d\sigma,$$

and the mutual potential energy becomes

$$W_1 = \int\int\int (V\rho') \cdot \frac{\rho^2 - \sigma^2}{8z} d\varphi d\rho d\sigma.$$

Taking account of both cases, $r < r'$, $r' < r$, we find that this formula gives W_1 when taken between the limits

$$\varphi = o \text{ to } 2\pi$$
$$\rho = z \text{ to } \infty$$
$$\sigma = -z \text{ to } z.$$

Performing one integration

$$W_1 = \frac{\pi}{4z}\int_z^\infty d\rho\int_{-z}^{z} d\sigma(V\rho')(\rho^2 - \sigma^2)$$

in a form readily applicable to electrons of any law of density. For the present case

$$\rho = \frac{e\lambda^3}{8\pi}\varepsilon^{-\lambda r'} = \frac{e\lambda^3}{8\pi}\varepsilon^{-\frac{\lambda}{2}(\rho - \sigma)}$$

$$V = \frac{e}{r}\left\{1 - \left(1 + \frac{\lambda r}{2}\right)\varepsilon^{-\lambda r}\right\}$$

$$= \frac{2e}{\rho + \sigma}\left\{1 - \left[1 + \frac{\lambda}{4}(\rho + \sigma)\right]\varepsilon^{-\frac{\lambda}{2}(\rho + \sigma)}\right\},$$

and thus

$$W_1=\frac{e^2\lambda^3}{16z}\int_z^\infty\int_{-z}^z(\rho-\sigma)d\rho d\sigma\left\{1-\frac{\lambda(\rho+\sigma)}{4}\varepsilon^{-\frac{\lambda}{2}(\rho+\sigma)}\right\}\varepsilon^{-\frac{\lambda}{2}(\rho-\sigma)}$$

$$=\frac{e^2\lambda^3}{16z}(I_1+I_2),$$

where

$$I_1=\int_z^\infty\int_{-z}^z d\rho d\sigma(\rho-\sigma)\varepsilon^{-\frac{\lambda}{2}(\rho-\sigma)}$$

$$I_2=-\frac{\lambda}{4}\int_z^\infty\int_{-z}^z d\rho d\sigma(\rho^2-\sigma^2)\varepsilon^{-\lambda\rho}.$$

Now

$$I_1=\varepsilon^{-\frac{\lambda z}{2}}\int_0^\infty\int_{-z}^z dud\sigma(z+u-\sigma)\varepsilon^{\frac{\lambda}{2}(\sigma-u)}\qquad(\rho=u+z)$$

$$=\varepsilon^{-\frac{\lambda z}{2}}\int_{-z}^z d\sigma\left\{\frac{2}{\lambda}(z-\sigma)+\frac{4}{\lambda^2}\right\}\varepsilon^{\frac{\lambda}{2}\sigma}$$

$$=\frac{2}{\lambda}\varepsilon^{-\frac{\lambda z}{2}}\left[\left\{\left(z-\sigma+\frac{2}{\lambda}\right)\frac{2}{\lambda}+\frac{4}{\lambda^2}\right\}\varepsilon^{\frac{\lambda}{2}\sigma}\right]_{-z}^z$$

$$=\frac{16}{\lambda^3}-\frac{4}{\lambda^2}\left(2z+\frac{4}{\lambda}\right)\varepsilon^{-\lambda z}$$

$$I_2=-\frac{\lambda}{2}\int_z^\infty d\rho\left(\rho^2z-\frac{z^3}{3}\right)\varepsilon^{-\lambda\rho}$$

$$=-\frac{\lambda}{2}\left[\left(\frac{z^3}{3\lambda}-\frac{\rho^2z}{\lambda}-\frac{2\rho z}{\lambda^2}-\frac{2z}{\lambda^3}\right)\varepsilon^{-\lambda\rho}\right]_z^\infty$$

$$=-\frac{\lambda}{2}\left(\frac{2z}{3\lambda}+\frac{2z^2}{\lambda^2}+\frac{2z}{\lambda^3}\right)\varepsilon^{-\lambda z}$$

$$=-\frac{z}{\lambda^2}\left(1+\lambda z+\frac{\lambda^2z^2}{3}\right)\varepsilon^{-\lambda z}.$$

And finally

$$W_1=\frac{e^2}{z}-\frac{e^2}{4z}(4+2\lambda z)\varepsilon^{-\lambda z}-\frac{e^2\lambda}{16}\left(1+\lambda z+\frac{\lambda^2z^2}{3}\right)\varepsilon^{-\lambda z}$$

is the mutual potential energy of the electrons, as a function of their distance z apart.

The force tending to increase z is $F=-\partial W_1/\partial z$, or

$$F=\frac{e^2}{z^2}-\frac{e^2\lambda^3z}{48}(1+\lambda z)\varepsilon^{-\lambda z}-\frac{e^2}{z^2}\left(1+\lambda z+\frac{1}{2}\lambda^2z^2\right)\varepsilon^{-\lambda z}$$

becoming e^2/z^2 at a great distance, as in the ordinary electron.

But at small distances the force does not tend to infinity, but to zero. We find

$$F\varepsilon^{\lambda z}=\frac{e^2}{z^2}\left(\varepsilon^{\lambda z}-1-\lambda z-\frac{1}{2}\lambda^2 z^2\right)-\frac{e^2\lambda^3 z}{48}(1+\lambda z)$$

$$=-7\frac{e^2\lambda^3 z}{48}\left(1+\frac{\lambda z}{7}\right)$$

when z is small in comparison with λ^{-1}. The force vanishes when the centres coincide, forming an electron of strength $2e$, as we should expect.

The exponentials are, for ordinary purposes, negligible even when z is a few multiples of λ^{-1}, which we may regard as the "radius" of the electron.

The electron $2e$ is essentially unstable, and e itself must be regarded as a constant of the aethereal structure, like λ. There is a probable relation between e and Planck's unit, as stated already.

Even when z is comparable with λ^{-1}, the force sinks to a small fraction of its value as given by the usual formula. For example, if $z=2/\lambda$, the force is only $0{\cdot}05e^2/z^2$.

This tendency of the force to vanish when the electrons are very close is, of course, applicable to electrons of opposite sign. But the inertia of the positive electron must be large, and this involves a large value of λ. We can show without difficulty that the inertia is proportional to λe^2 for slow speeds, and thus λ for the positive electron, if such exists, must be of the order of 1,000 times the value of λ for a negative electron.

But the practical evanescence of the force will remain, and there is even the possibility that the sign of the force may change, with a suitable law of density. Thus a positive and negative electron would not necessarily rush together and annihilate each other, but would form a doublet, whose length would be comparable with the radius of a single electron. Even if the force did not change sign, so that the positive and negative elements had no position of relative equilibrium, an oscillation about each other in simple harmonic motion is possible. For example, if the values of λ were equal, the electrons being of the above type, the equation for the distance z between them would be

$$\frac{m\ddot{z}}{2}=-\frac{7e^2\lambda^3 z}{48},$$

where m is the mass of either. Since m is of order λe^2, the

system would emit a wave-length comparable with the "radius" of an electron.

If the doublet is endowed with a rotation, it can preserve a constant length, and the present investigation is given merely as an illustration of the possibility of such doublets.

The view that two aethereal structures can exist in this way without deformation in presence of each other, and simultaneously occupying the whole aether, is, of course, difficult. But the difficulty is no greater than that of postulating the ordinary bounded electron, each of whose parts must repel each other. If actual deformation takes place, it is not apparently possible to find a basis of calculation without further hypotheses incapable of verification, so that the present suggestion of the possible nature and existence of doublets is sufficient for the purpose in view. It is, perhaps, worthy of remark that the Lorentz formula for mass as a function of velocity can be obtained for this type of electron, with λ^{-1} substituted for the radius. The whole distribution of density ρ may be treated as in the principle of relativity. It does not seem necessary to give the complete analysis.

From this point of view a neutral doublet could consist of a distribution of density of the form

$$\rho = \frac{e}{8\pi}(\lambda_1^3 \epsilon^{-\lambda_1 r} - \lambda_2^3 \epsilon^{-\lambda_2 r})$$

around the origin as centre. The total charge is zero when integrated throughout space, and the density and electric force are confined to a region round the origin whose radius is of order λ_1^{-1} $(\lambda_1 < \lambda_2)$ almost precisely. The system behaves like a pair of charges $\pm e$ at the origin, of radii λ_1^{-1}, λ_2^{-1}, which can be separated by the application in a suitable form of an amount of energy of order $\lambda_1 e^2$.

Such a type of doublet, whose charges were any multiple of that of an electron, might form a component of a complex atomic nucleus, and radioactivity would then occur when it was partially dissolved into component α and β particles. It would appear to be necessary to consider a theory of this type, in view of the extreme smallness of the nuclei of atoms, as determined by Rutherford and others, which nevertheless possess the property of evolving large numbers of α and β particles, the latter being of the same order of size as the nucleus itself.

In again laying emphasis on the fact that the present note is only intended to be a suggestion towards a point of view, rather than a development of the view which could serve no useful purpose at the present stage—Sir Ernest Rutherford has proposed to leave nuclear structure to the next generation—it is to be noticed that the difficulties inherent in the permanent fixation of these types of strain, capable of superposition, are not greater than those involved in the internal equilibrium of the mutually repelling parts of the ordinary bounded electron.

ABSTRACT.

The electron is usually regarded as a globule of electricity with a definite radius. This conception has proved valuable, but involves difficulties in connection with the nuclear structure of complex atoms. On the view that the electron consists of a region of strain in the æther such line constants should have some significance throughout the whole æther; which may, in fact, be in some manner cellular, with these linear magnitudes involved in the specification of the cells, and therefore in any strained structure composed of them.

The electron would be regarded as a state of strain which for practical purposes is concentrated at its centre, rapidly diminishing outwards according to some very convergent law involving some line constant in its specification. By way of illustration the idea is worked out mathematically on the assumption that the strain varies as $e^{-\lambda r}$, on which hypothesis λ^{-1} is the "radius." It can be shown that the Lorentz formula for mass as a function of velocity can be obtained for this type of electron. The cha ge on the electron is regarded as a fundamental property of the æther, and is related to Planck's constant h.

DISCUSSION.

Dr. H. S. Allen: There can be little doubt of the existence of a relation, referred to by Prof. Nicholson, between Planck's constant h and the charge of an electron e. The relation suggested by Lewis and Adams may be written—

$$ch=\sqrt[3]{\frac{8\pi^5}{15}}(4\pi e)^2,$$

where c is the velocity of light. Taking Millikan's latest value for e $(4{\cdot}774\times10^{-10})$ and $c=3\times10^{10}$, we find $h=6{\cdot}558\times10^{-27}$. From his photoelectric experiments Millikan found $h=6{\cdot}57\times10^{-27}$ within about 0·5 per cent., and in his latest table of fundamental constants he gives $h=(6{\cdot}547\pm0{\cdot}013)\times10^{-27}$. Thus the agreement is within the limits of experimental error. All the principal radiation constants can be expressed in terms of e. The curious numerical relations between the primary constants of physics, to which attention was directed in my Paper read before the Society in 1915, depend upon the above formula connecting h and e. On the lines suggested in Prof. Nicholson's Paper it would seem as if most, if not all, of the important constants of nature may be referred to some fundamental property of the æther.

Sir Oliver Lodge (communicated): I am much interested in Prof. Nicholson's ingenious plan for doing away with the definite boundary of

an electron, and devising a mathematical scheme which shall enable us to regard it as a point-centre of strain decreasing exponentially in every direction without limit, so that the linear dimension associated with it shall be—like many time-constants—the distance at which the density is reduced to $\frac{1}{e}$th of what it is at the centre. This plan, if it can be developed properly, seems to get over many of the difficulties about the coherence of parts of a charge, and about the extraordinary properties of a nucleus, which though, from some points of view, an extremely small and highly-charged unit, yet necessarily has a complexity which enables it to be disintegrated and fired off in fragments. The ready permeability or inter-penetrability without destruction, of Prof. Nicholson's conception of an electric unit, seems likely to diminish the difficulty of conceiving such a nucleus; and on the whole his suggestions seem to me helpful and valuable. I do not feel justified in saying more at the present time.

Prof. Nicholson, in reply, said it was of interest to see that Millikan's final value of h was practically equal to the first value that had been obtained for that constant.

II. *On the Thermo-Electric Properties of Fused Metals.* By CHARLES ROBERT DARLING *and* ARTHUR W. GRACE.

RECEIVED, OCTOBER 25, 1917.

METALS WITH MELTING POINTS BELOW 700°C.

IN a previous Paper on this subject ("Proceedings," Vol. XXIX., Part I.) an account was given of the experimental methods employed in investigating the behaviour of bismuth up to 560°C., which was the highest temperature attainable with the apparatus described. As the object of the research was to test the possibility of using a couple of one or two fused metals for measuring temperatures, it was necessary to resort to a new method of procedure in order to carry the observations with metals in general into the region above 600°C. to as high a limit as possible. The present Paper deals with the new experimental methods, and the results obtained with a number of metals up to 1,000°C.

Experimental.

Preliminary trials were made with a silica tube, of about 2·5 cm. inner diameter, closed at one end, and wound externally with a nichrom wire through which a current was passed. This heating element was lagged with magnesia, and the metal under test dropped into the tube and melted in situ. A wire of the second metal composing the couple, protected by a graphite tip, was inserted in the liquid mass, and side-by-side with this a pyrometer was placed. An overflow of the fused metal along a sheet of uralite was arranged as described in the previous Paper, the cold junction being formed at the end of the overflow. It was found, however, that after one or two heatings the silica tube was cracked, and as a second trial ended similarly, the method was abandoned in favour of others to be described, and illustrated in Figs. 1 and 2. In the first of these methods (Fig. 1) a silica tube *B*, open at both ends, was inserted about halfway up the vertical tube *A* of an electric furnace (nichrom wound), whilst the lower end of *B* dipped into a vessel containing oil. Prior to insertion in the furnace, *B* was filled with the metal under trial, and the second metal placed in the liquid mass in the form of wires, forming the hot junction *H* and the cold junction *C*. After placing in the furnace the upper part of the metal in *B* was

melted, the pyrometer P inserted, and the hot and cold junctions coupled to a calibrated galvanometer G. Readings of E.M.F. were taken at suitable temperature intervals, the furnace being controlled by an external resistance. This method enabled satisfactory readings to be obtained with antimony, which expands on solidification, as the top surface remained liquid while solidification was proceeding below, thus permitting of free expansion. The second method was found

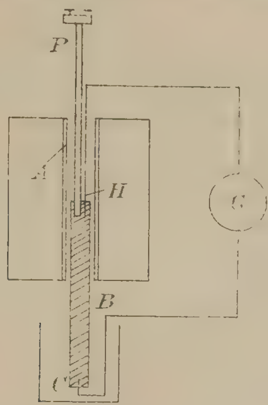


Fig. 1.

to be more convenient than the foregoing in the case of metals procurable in the form of long rods. A graphite block, G, Fig. 2, was bored with two holes, into which were inserted tight-fitting silica tubes, A and B, 40 cm. long, each containing slack-fitting rods of the metals under examination. When one of the metals was known not to fuse at the temperature attained

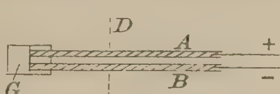


Fig. 2.

in the experiment the end was threaded and screwed into the graphite. This element was inserted in the tube of an electric furnace and used in the horizontal position, wires of the same material being taken from the cold end to the measuring apparatus. When placed in the furnace to a distance indicated by the line D, continuity of the circuit was maintained between the liquid portion in the furnace and the solid part beyond D.

Temperatures were measured by means of a thermo-couple of Hoskins' alloys, kindly provided by the Foster Instrument Co. This couple was inserted in a cavity in the graphite block *G*, and in taking readings the furnace resistance was adjusted at intervals, and observations made when the pyrometer indicator and also the indicator connected to the metals under test were both stationary. The results obtained with a number of metals are appended, and refer to a cold-junction temperature of 25°.

Lead.

A large number of observations were made with lead, with a view to testing the validity of extrapolation in the thermo-electric diagram as well as to discover the effect of fusion. It was found that no abrupt change is produced on melting, all

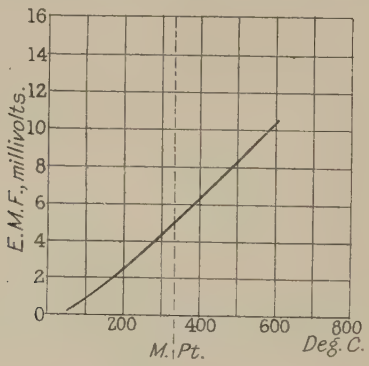


FIG. 3.

the curves obtained by plotting E.M.F. against temperature being quite smooth in this region. It was also found that whereas up to about 300°C. all these curves were approximately parabolic, the continuations above this temperature departed considerably from this shape. As an example of this, reference may be made to Fig. 3, which represents the values obtained with a german-silver-lead couple. There is no discontinuity at 327°, the melting point of lead, but from about 300° upwards the temperature—E.M.F. relation becomes linear, and consequently $\frac{dE}{dT}$ has a constant value over this range. The representation of this couple on a thermo-electric diagram would, therefore, consist of a sloping line up to 300°, and afterwards of a horizontal line parallel to the lead axis. It is evident,

therefore, that extrapolation of the sloping lines obtained from low-temperature observations leads to serious errors, and many published diagrams are quite erroneous for this reason. As an example, it is specifically stated in some books that the neutral point of iron and lead is 350°C. beyond which the E.M.F. diminishes; whereas direct observation shows that no such neutral point exists, and that the E.M.F., which is 2·6 mv. at 350°, rises continuously to a value of 4·2 mv. at 900°. A correct thermo-electric diagram for various metals up to 1,000°C. has yet to be prepared; and as such would no longer possess the simple character of one based on a parabolic relation between E.M.F. and T, it would be of doubtful value.

Tin.

As the high boiling point of tin renders it feasible for use as one of the members of a liquid junction, many experiments

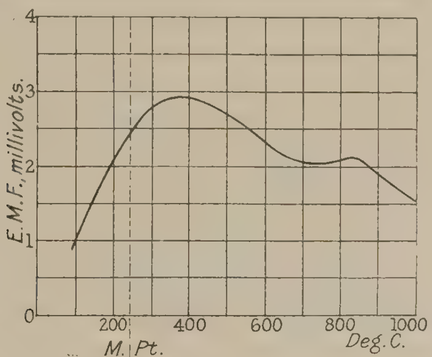


FIG. 4.

have been made with this end in view. In no instance was any discontinuity noticed at the melting point (232°C.), as indicated in Fig. 4 (iron-tin), in which the shape of the curve in this region is well suited to the detection of even a slight change. This curve also illustrates the dangers of extrapolation, based on the usual assumption that the curve beyond the neutral point will be a geometric continuation of the earlier portion. As will be seen, the steepness diminishes considerably after passing the neutral point and between 700° and 850° shows a flexure, after which the steepness again increases. This behaviour of iron in the recalescence region was first noticed by Belloc,* and is well shown when coupled with tin,

* Ann. de Chim. et de Phys., 30, p. 42, 1903.

although obscured in the case of an iron-constantan junction, when the E.M.F. under measurement is 20 times as large, entailing the use of a much coarser indicator.

Cadmium, Zinc and Aluminium.

In the case of these three metals it was also noticed that the act of melting produced no change in the E.M.F. The temperature of inversion of zinc and iron was observed to be about 470°, which is about 50° above the melting point, and had any change resulted from fusion it would have been detected readily with this couple. Aluminium and constantan show a linear relation between E.M.F. and temperature, which is not interrupted by fusion at 657°.

Antimony.

Special interest was attached to the experiments with antimony, owing to the general resemblances between this metal and bismuth. After several unsatisfactory attempts with

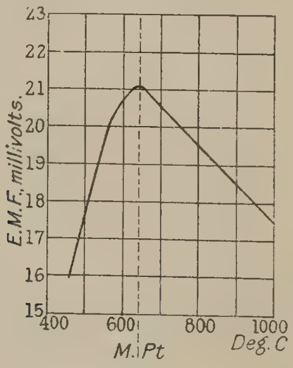


FIG. 5.

different arrangements, a successful set of readings were obtained with the apparatus shown in Fig. 1. It was found that, as in the case of bismuth, an abrupt change in thermo-electric properties occurred at the melting point, 632°. A typical case is shown in Fig. 5, which shows the behaviour of an antimony-copper couple between 400° and 1,000°, the change in shape due to fusion being most pronounced. A similar result was obtained with an antimony-iron couple, and hence antimony acts in the same manner as bismuth in this respect.

Conclusions.

So far as it is possible to generalise on the results obtained, it would appear that the thermo-electric properties of metals are usually unaffected by change of state from the solid to the liquid phase or vice versa. The exceptional behaviour of bismuth and antimony may be due to the formation of allotropic modifications on melting, in support of which view may be adduced the fact that both of these metals expand upon solidification, and are thus exceptions to the ordinary rule. Further, as shown in the case of iron, an allotropic change is accompanied by an alteration in thermo-electric properties, and it is possible that the one change is always accompanied by the other. Experiments are now in progress with metals of still higher melting points, which may confirm or otherwise the view expressed above. The success of the main object of the research—the production of a high-reading pyrometer—entails the condition that mere change of state has no effect on the thermo-electric properties of the metals used.

It is possible that molecular changes occurring in molten alloys may be detected by experimental methods similar to those described in the present communication. A suitable metal to couple with the alloy under test could be found by trial, and it is probable that the change in E.M.F. accompanying a molecular transformation would in some cases be detected with greater certainty than a small temperature halt. This is a matter which it is hoped to investigate later, as opportunities permit. Experiments are also desirable in which both metals forming the couple undergo liquefaction, which it is also hoped to conduct at some future time.

We would point out that the values of E.M.F. given in the present and the previous Paper refer only to our samples of metals, which were purchased without any specification as to purity. As is well known, different specimens of what are reputed to be the same metal frequently vary in thermo-electric properties; thus two pieces of platinum wire from different sources usually show an E.M.F. when joined and heated. We have, therefore, preferred in all cases to take a direct observation, rather than to work with a single metal and deduce the other results, as suggested by A. Campbell in a criticism of the previous Paper. We find that we have thus saved ourselves from many errors, particularly in the case of alloys such as constantan and nichrom, which vary consider-

ably in composition, but are nevertheless very valuable in thermo-electric work.

ABSTRACT.

In a previous Paper ("Proceedings," Vol. XXIX., Part I.) the authors described experiments with bismuth, the apparatus then used only being capable of furnishing readings up to 560°C. Methods have now been devised in which the metals examined may be heated in the tube of an electric furnace, and observations made up to the temperature limit of the furnace. The metals experimented with were lead, tin and antimony up to 1,000°C., and zinc and cadmium up to temperatures approaching the boiling point. No change in thermoelectric properties was noticed at fusion, except in the case of antimony, which, like bismuth, shows an abrupt bend in the E.M.F.-temperature curve at the melting point, 632°C. This exceptional behaviour of antimony and bismuth is in keeping with the anomalous properties of these metals, both of which expand on solidification; and it is suggested that an allotropic change occurs at fusion in these metals.

In the case of lead which is used as the reference metal in thermo-electric diagrams, it is shown that extrapolation of lines in the diagram beyond 300° led to serious errors, and that although at low temperatures the E.M.F.-temperature curves are approximate parabolas, the departure from this shape above 300° is so marked as to render thermo-electric diagrams of little value.

DISCUSSION.

Mr. Whipple said that this work opened up certain possibilities of commercial importance, as it appeared that information could be obtained of the thermo-electric properties which a particular alloy would have while it was still in the molten state. It would thus be possible by adding one or other of the constituents as required to obtain an alloy with any prescribed thermo-electric properties. At present the alloy had to be allowed to cool, and a wire of it drawn and tested. If it were not right, it had to be melted up again and its constitution altered, which was a troublesome method. With regard to the high boiling point and low vapour pressure of tin, it was of interest to observe that Northrup had suggested a tin graphite thermometer for high temperatures up to about 1,700°C. on the same lines as the ordinary mercury in glass thermometers. The tin expanding along the stem of the thermometer moved an index wire by which the temperature was indicated. What was the magnitude of the change in properties of the iron-constantan couple on passing the recalescence point of iron? He was very interested in this point, as he had not noticed any such change himself.

The President said it was a useful thing to have the futility of the old thermo-electric diagram proved so thoroughly.

Dr. Willows suggested that a zinc-mercury amalgam would be an interesting substance to examine thermo-electrically for allotropic change-points. Its resistance curve between 0° and 100° shows marked evidence of such changes.

Mr. Darling, in reply, said he had not detected the recalescence change with the iron-constantan couple probably because of the relatively large total E.M.F. of that couple. The change was easily detected, however, with the iron-tin couple, since the total E.M.F. is then only about 3 millivolts and a delicate galvanometer is used.

III. *Triple Cemented Telescope Objectives.* *By* T. SMITH, *B.A., and* Miss A. B. DALE. (*From the National Physical Laboratory.*)

RECEIVED OCTOBER 18, 1917.

THE ordinary telescope objective consists of a crown lens and a flint lens which are not cemented together because the conditions which must usually be satisfied demand a difference in the curvatures of the inner surfaces of the two lenses. For some purposes it is desirable to have a cemented objective, even if this involves some falling off in the quality of the definition obtainable. There are, however, limits beyond which such deterioration may not go, and these depend on the relative aperture and the field of view of the lens. Unfortunately the circumstances in which a cemented objective is required are usually those which also involve large relative apertures and a large field of view. If a triple cemented objective is substituted for a doublet the extra degree of freedom obtained enables the required conditions to be satisfied much more nearly than in the case of the simpler form of objective, even though no additional variety of glass is used for the third component lens. The extent of the advantage thus gained depends upon the magnitude of the outstanding aberrations and upon the magnitude of the curvatures required for the triple objective. A manufacturer will require to have shallower surfaces to grind as a compensation for the additional labour involved in the introduction of two extra surfaces into the optical system, and the performance of the telescope will not be sufficiently improved if the reconciliation of the conditions for the removal of spherical aberration and coma of the first order involves the introduction of greatly increased aberrations of the second order. The object of the investigation described in the present paper is to determine what two kinds of glass should be used in order to obtain a triple telescope objective that is most satisfactory as regards smallness both of curvatures and of second order aberrations.

The commonly accepted opinion is that these two characteristics go hand in hand, so that among otherwise equally good objectives that one will have the smallest second order aberrations which has the least curvatures. This, no doubt, may be taken as a rough guide, but it will be seen from the results detailed below that it is not strictly true, for the series of triple objectives which are best as regards smallness of second

order aberrations have somewhat greater curvatures than another series of triplets which are not quite so satisfactory in this respect.

In the calculation of all objectives thicknesses have been entirely neglected. This course enables the amount of numerical work involved to be reduced to the minimum without causing errors of any consequence in the conclusions drawn from the results of the investigation. A further simplification has been introduced by assuming the refractive index of the flint glass to be 1·6200 throughout the whole series of objectives. With this one exception the results are quite general, as both the refractive index of the crown glass and the ratio of the dispersions produced by the two glasses have been varied over a sufficiently wide range to embrace all the varieties of glass that are likely to be employed for the construction of telescopic objectives.

The objectives considered have in all cases been calculated so that all first order spherical aberration and coma is removed. These are not the conditions that are satisfied in actual objectives, a balance between first and second order aberrations of opposite signs being decidedly preferable. It might be supposed that this would detract from the utility of the investigations; this is, however, not the case. The curvatures of the actual objectives in which such a balance is obtained are almost identical with those of thin objectives free from first order aberrations, and the differences in the magnitude of the second order aberrations must, therefore, be small if the actual objectives are of normal thickness.

The formulæ from which triple objectives are to be calculated have been given in a previous paper by one of the authors.* The triplet is most simply regarded as two doublets cemented together; the aberrations of a triplet then depend upon :—

(*a*) The minimum spherical aberration for magnification -1 for the doublet, the corresponding amount of coma, and the curvatures of the external surfaces;

(*b*) The amount of spherical aberration when a plane wave is incident normally on a doublet having its first surface plane;

(*c*) The Petzval sum.

These quantities are plotted in Figs. 1 to 4, for a varying refractive index of the crown glass and for ν ratios of crown to

* "Notes on the Calculation of Thin Objectives." Proc. Phys. Soc., Vol. XXVII., p. 495.

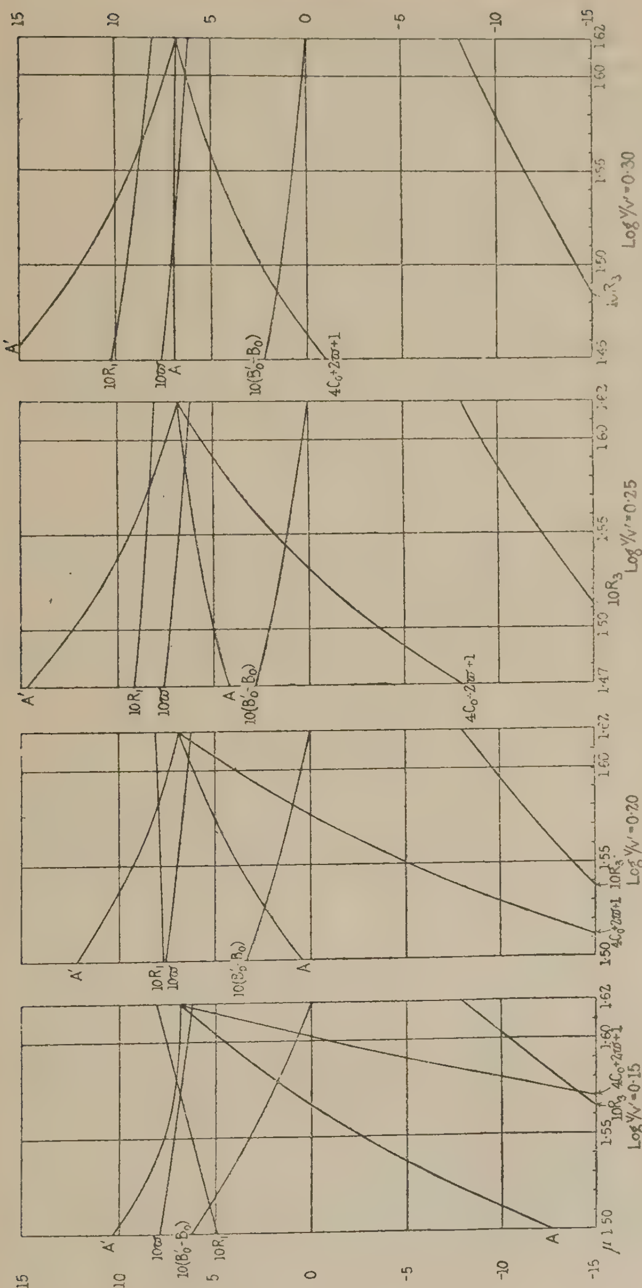


Fig. 1.

Fig. 2.

Fig. 3.

Fig. 4.

Fundamental Constants of Doublets.

flint of $10^{0\cdot15}$, $10^{0\cdot20}$, $10^{0\cdot25}$ and $10^{0\cdot30}$.* For a doublet with a flat outer crown surface the spherical aberration is A for an object at infinity. When a plane wave is first incident on a flat outer flint surface the spherical aberration is A'. R_1 and R_3 are the outer curvatures of crown and flint components respectively of the doublet which has minimum spherical aberration for magnification -1, and this spherical aberration is denoted by $4C_0+2\varpi+1$. $B_0'-B_0$ is the corresponding amount of coma in this objective, and ϖ is the Petzval sum. A knowledge of the way in which these quantities depend upon the difference in the refractive indices of crown and flint glasses and upon their ν ratio is of the greatest importance to the designer of optical systems.

The features of the diagrams which are of most consequence are the following :—

(a) A' is always positive, and the A' curve varies very little in Figs. 2, 3 and 4.

(b) A is practically constant for all refractive indices of the crown component in Fig. 4, but decreases rapidly as the ν ratio is diminished.†

(c) $4C_0+2\varpi+1$ diminishes with a reduction of either the crown glass index or the ν ratio. When both refractive indices become equal, A, A' and $4C_0+2\varpi+1$ take the common value $\left(\frac{\mu}{\mu-1}\right)^2$.

It has already been pointed out that the simultaneous fulfilment of the conditions for freedom from spherical aberration and coma in a cemented doublet is dependent upon the choice of a suitable combination of glasses. The classification of triple objectives may be conveniently based upon the cemented doublets which satisfy the same conditions. For a given ν ratio there is some one refractive index for the crown glass which will enable a doublet with the crown component leading to be made free from spherical aberration and coma. If now the crown refractive index is slightly altered a doublet can be made to satisfy one condition, while departing slightly from the other. To satisfy both conditions the doublet must

* If the objective is not to be corrected exactly for chromatic aberration these figures represent the absolute value of the ratio of the power of the crown lens to that of the flint.

† Glasses for which A vanishes are those suitable for the construction of symmetrical triplets for magnification -1. Since A' is always positive there is no possibility of a corresponding symmetrical form with external crown lenses.

be replaced by a triplet, and obviously this triplet will only depart slightly from the doublet form, the additional component being a very weak lens. Evidently two triplets can be found to satisfy the conditions, one consisting of a weak crown component added after the flint lens, and another with a weak flint component placed in front of the crown lens. As greater variations in the refractive index are introduced the departures from the doublet form will become more pronounced, but over an appreciable range of indices it is to be expected that the derivation of these two triple objectives from a doublet with the crown component leading will be evident. In a like manner two series of triple objectives may be derived from a doublet with the flint component leading. The classification adopted here will therefore be as follows :—

Series I. and II.—External Lenses of Crown Glass.

Series I. based on doublet with crown component leading, and therefore tending to have the first crown component more powerful than the second.

Series II. based on doublet with flint component leading, and therefore tending to have the first crown component weaker than the second.

Series III. and IV.—External Lenses of Flint Glass.

Series III. based on doublet with crown component leading, and therefore tending to have the front flint component weaker than the second.

Series IV. based on doublet with flint component leading. and therefore tending to have the front flint component more powerful than the second.

The curves showing any properties of Series I. and III. must necessarily intersect at the position relating to the doublet. since this is a member of both series. Similarly, curves relating to II. and IV. must intersect at the position of the second doublet. It is to be noted that there is no reason why the curves showing certain properties of these allied series should not also intersect at additional points.

Each series is divided by the double points into two parts, and the character of the triplet in these sections is quite distinct. The distribution of the total power of the lenses made of the external glass between the two components varies progressively in the series. and at the double points the power of one of these components becomes zero. If. then, the powers of

both external components are of the same sign on one side of the double point they will necessarily be of opposite sign on the other side. It is to be expected that the curvatures and higher order aberrations of any one series will be greater in absolute value on the latter side of the double point.* This side, it will be seen, is that on which the difference of refractive indices of the two glasses is less than for the doublet. From results previously established for double objectives * it follows that the normal glasses now available lie in the region more favourable for small curvatures and higher order aberrations.

If in any region the solution for a series becomes imaginary, the solution for another series must also have the same property. The series associated in this way with one another are obviously those which have the same glass externally, since in such a case the one series must be a continuation of the other. It is found that the solutions to Series I. and II. are always real, but those of Series III. and IV. become imaginary when the difference between the refractive indices exceeds a certain amount depending upon the ν ratio.

Figs. 5, 6, 7 and 8 show the second order aberration coefficients for triple objectives calculated from the data given in Figs. 1 to 4. The full curves show the amount of spherical aberration and the dotted curves the departure from the sine condition. The heavy lines relate to Series I. and III., which approximate to a doublet with the crown component leading, and the lighter lines to Series II. and IV. It will be noted that these two outstanding quantities are always negative. Thus the spherical aberration of the second order always tends in the direction of over-correction. The coma depends upon the difference between the full and the dotted curves, and is thus of one sign for Series I. and III. and of the opposite sign for Series II. and IV. Generally speaking, the amount of coma does not vary very greatly between the various series except near the junction of Series III. and IV., where the coma is small for systems with external flint components. As would be expected from a knowledge of the second order aberrations of achromatic doublets, the spherical aberration is distinctly less for Series I. than for Series II., and for Series III. than for Series IV. The most interesting feature of these diagrams, however, is the way in which the curves for combinations with external flint lenses lie above those with external crown lenses,

* *See* "The Choice of Glass for Cemented Objectives." Proc. Phys Soc., Vol. XXVIII., Figs. 1 and 2.

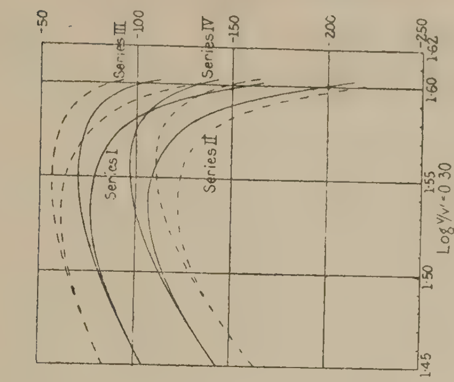


FIG. 8.

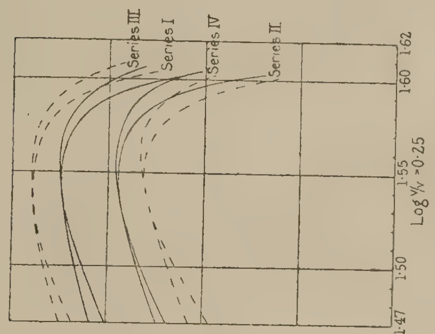


FIG. 7.

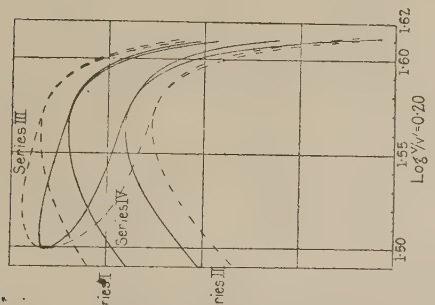


FIG. 6.

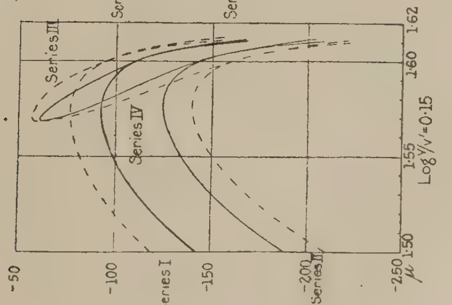


FIG. 5.

SECOND ORDER SPHERICAL ABERRATION AND COMA.

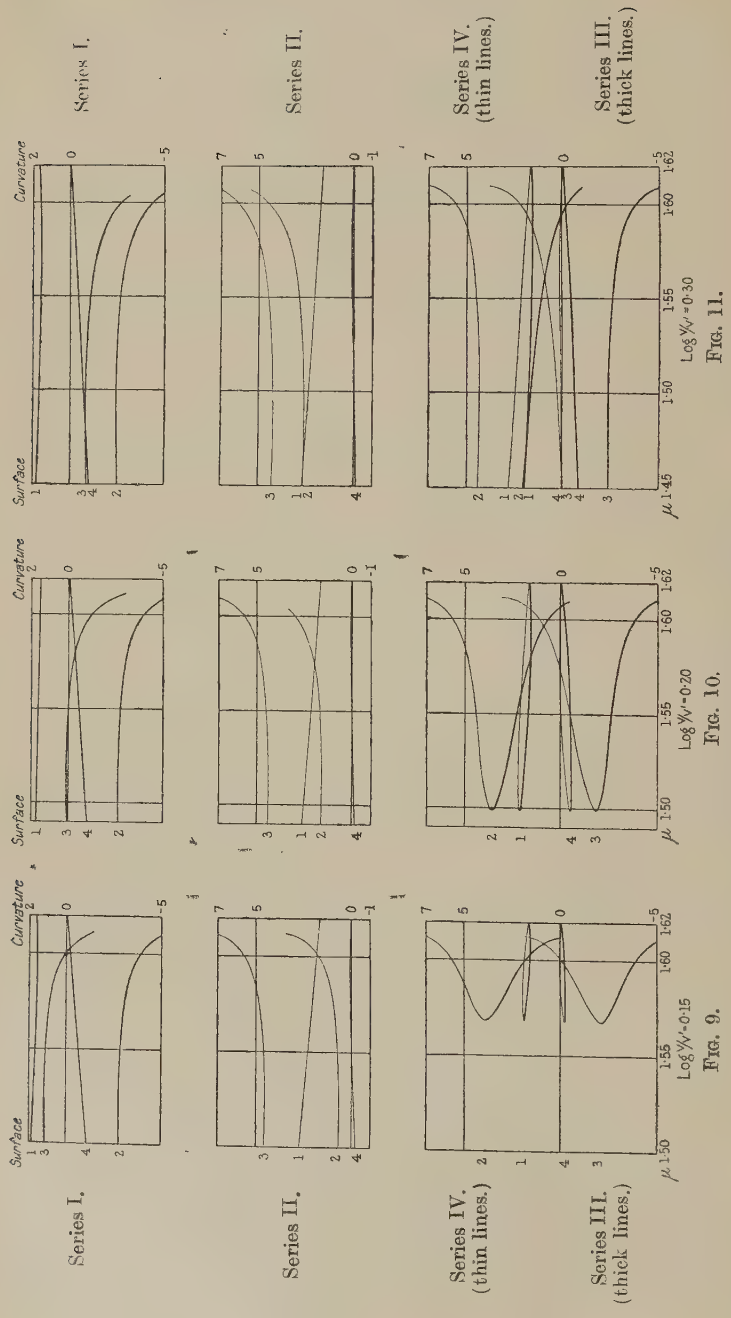


FIG. 9.

FIG. 10.

FIG. 11.

CURVATURES OF TRIPLET SURFACES.

Series III. thus being the best as regards spherical aberration. It is also remarkable that the minimum amount of this aberration left outstanding should vary so slightly over the range of ν values covered by the diagrams.

It was originally intended, when this investigation was undertaken, to plot in a diagram, in which the refractive index of the crown glass and the ν ratio were the abscissæ and ordinates respectively, contours showing for which glasses the aberrations attained equal values. The four figures shown, however, indicate that this would involve an amount of labour which the results would not justify. For the contours are not in all cases simple closed curves, but are broken up into at least two parts, as may be seen by noting that in Figs. 5, 6 and 8 the curves cross the line -60, but do not reach it in Fig. 7. The approximate constancy of the position of the highest points of the curves removes the need for accurately plotted contours of the character at first proposed.

The remaining figures show the curvatures of the surfaces. The curvatures of the external surfaces vary between comparatively close limits, and are chiefly determined by the condition for freedom from coma. The curves giving the curvatures of the two cemented surfaces are approximately parallel to one another. The inner surfaces tend to become very strongly curved when the refractive index of the crown lens approaches that of the flint. In Series I. and III., represented by the thick curves, the greatest curvatures are distinctly less than for Series II. and IV., shown by thin lines. In the former two series the cemented surfaces become concave to the incident light, and in the latter two series convex, when the refractive index of the crown glass is only slightly less than that of the flint. The curvature of the steepest surface in Series I. is sometimes greater and sometimes less than in Series III.; but the curvatures of Series I. are decidedly the more favourable when the outstanding aberration of Series III. is least. In the case of greatest practical importance the two conditions are therefore inconsistent. The triple objectives of both Series I. and III. show decided advantages over most doublets, both as regards smallness of curvature and smallness of outstanding spherical aberration. Considerable gain is thus possible by their use for many purposes where the conditions are distinctly difficult of fulfilment with an objective of the ordinary type.

Figs. 9, 10, 11 suggest that for small variations in the μ and ν values of the glasses the principal alterations in the curvatures should usually be carried out on the surfaces indicated in the following table :—

Series.	Variation in μ.	Variation in ν.
I.	4	3
II.	1	2
III.	2 and 3	2
IV.	3 and 2	3

In all cases a variation in ν is chiefly compensated by an alteration of an internal surface. In lenses with exterior crown components alterations for both μ and ν should be carried out on the weak correcting component. The fact that each correction can be carried out on a single surface constitutes an important manufacturing advantage for these two series. If Series I. is adopted, and the only factor likely to vary appreciably is the refractive index of the glass, when objectives of a given focal length are required it should be possible to use with all glass meltings the same standard tools for all the surfaces excepting the last.

ABSTRACT.

The Paper describes the four series of triple cemented thin telescope objectives which can be made from two kinds of glass, and determines their construction when first order spherical aberration and coma are eliminated. The second order spherical aberration and coma are then calculated, and the former found to be of the same sign for all optical glasses when the surfaces are spherical. The best standard attainable varies very little over a considerable range of glasses. Diagrams show the variations in the curvatures as the glasses are varied for refractive index and dispersion. Contrary to the general belief, it is found that the objectives with least second order aberrations (absolute values) are not those with the least curvatures for their refracting surfaces.

IV. *On a Class of Multiple Thin Objectives.* *By* T. Smith, B.A. (*From the National Physical Laboratory.*)

Telescope objectives usually consist of two component lenses mounted in contact, and their optical properties differ very little from those of infinitely thin lenses. An objective consisting of only a single piece of glass may be made free from spherical aberration and from coma for light of a specified wave-length, but its surfaces will in general not be spherical. By using two components made from glasses of suitably differing optical properties, the chief defect of the single lens, the large differences in the position of the focus with light of various colours, may be replaced by a much less serious fault. With a single lens the distance of the focus from the objective invariably increases as the wave-length of the light is increased. When two kinds of glass are employed this distance may be made first to decrease, then become stationary, and afterwards to increase; the rates of both increase and decrease under these conditions are many times less than the corresponding rate of increase in the single lens. The wave-length for which the position of the focus is stationary is carefully selected according to the purpose for which the objective is to be used. When this wave length is given the relation between the positions of the focus for light of various wave-lengths is quite definite, and depends only upon the kinds of glass of which the objective is made. No matter how the shapes of the lenses may vary, all the objectives of the same two glasses and of the same focal length will have identically the same relation between the position of the focus and the wave-length of the light if the focus is stationary for the same wave-length in every case. It follows that any conditions relating to the chromatic aberration of the focal distance must be met by a suitable choice of the kinds of glass of which an objective is to be made, and that the shapes of the lenses may be arranged to satisfy other aberration conditions.

When two lenses are employed it is found that the two remaining conditions which it is most important to satisfy, freedom from spherical aberration and from coma for light of a given colour, can be satisfied, though the four surfaces are restricted to the spherical form. This being so, objectives have almost invariably been designed with spherical surfaces to the exclusion of all other forms, on account of the great

advantages the former offer in ease of manufacture and testing in comparison with the latter. It is important to realise that the difficulties of manufacturing non-spherical surfaces to the extremely high degree of accuracy essential in optical work have been the decisive factor. The calculation of lenses having surfaces of other forms offers little difficulty. In fact, the calculation of such optical systems as photographic lenses, where many conditions must be approximately satisfied at the same time, would be very greatly simplified if the restriction to spherical surfaces were removed.

The statement that the removal of spherical aberration and coma from a doublet lens is consistent with the employment of spherical surfaces needs some qualification. It is strictly true if by spherical aberration and coma, first order spherical aberration and first order coma are meant. These are much the most important monochromatic aberrations in telescope objectives, but the corresponding aberrations of the second and third orders are appreciable, particularly in objectives of large relative aperture. The conditions for the removal of these residual aberrations are not compatible with the absence of the first order aberrations when the surfaces are strictly spherical, and such an objective will consequently bring the light which has traversed various zones of the lens aperture to somewhat different foci. In large telescope objectives the residual aberrations are reduced by deliberately departing from the spherical form for one or more of the lens surfaces. This "figuring" process, as it is called, is much too costly to be applicable to lenses that are required in large numbers, but it fortunately happens that the need of figuring in the case of these smaller lenses is comparatively rare. There are nevertheless many cases in which the removal of the zonal aberrations would lead to an appreciable improvement in the performance of the optical system to which the objective belongs.

In many instruments in which thin objectives are employed the total number of lenses is inevitably considerable, and it is important to avoid the introduction of unnecessary glass-air surfaces on account of the loss of light which each involves. Wherever possible the components of an objective should have equal curvatures on successive internal surfaces, so that these may be cemented together, and the losses by reflexion reduced. When the objective is a cemented doublet two aberration conditions can only be satisfied by making choice of the proper

kinds of glass.* With a triplet using only two kinds of glass two aberration conditions can always be satisfied.† As a rule, four triplets can be found to satisfy the conditions. The cemented triplet has the same number of degrees of freedom and the same limitations as the uncemented doublet ; that is to say, the satisfaction of the first order aberration conditions is inconsistent with the absence of higher order aberrations when the surfaces are spherical‡. One of the simplest ways in which the residual aberrations might possibly be removed consists in the employment of only two kinds of glass, but with at least two lenses of each glass, the condition that all the surfaces are to be spherical, being, of course, retained. The present paper deals with a method by which such multiple lenses can be calculated. A series of lenses consisting of four thin components has been calculated by the method described, and their second order spherical aberration coefficients determined. The number of lenses of this kind so far investigated is too small to enable the maximum advantages which may be secured to be estimated. The series examined, however, shows that the ratio of the shortest radius to the focal length may be very much greater than in simpler lenses satisfying the same first order conditions, and suggests that it may be possible to secure this advantage in a lens containing surfaces of only three different radii.

In addition to the defects already mentioned there is another which is often serious. When the spherical aberration has been corrected for light of one wave-length the spherical aberration for sensibly different wave-lengths is far from negligible. The method of calculation adopted enables the chromatic differences of first order spherical aberration and first order coma to be determined at once in terms of those of a cemented achromatic doublet of the same two glasses. It is not necessary for this purpose to complete the determination of the compound lens.

The method employed is an extension of that already adopted for the calculation of triple objectives. The triple objective is regarded as a combination of two cemented achromatic doublets, the external surfaces of one kind of glass

* "Notes on the Calculation of Thin Objectives." Proc. Phys. Soc., Vol. XXVII., p. 498.

† "The Choice of Glass for Cemented Objectives." Proc. Phys. Soc., Vol. XXVIII., p. 232.

‡ "Triple Cemented Telescope Objectives." Proc. Phys. Soc., Vol. XXX., p. 21.

being of equal and opposite curvatures, so that when the two doublets are put together no air lens is left between them. It may be supposed that n achromatic doublets are to be combined in the present instance. Let the typical doublet of unit focal length have its crown component of power K, and its flint of power K'. Then

$$K+K'=1.$$

Take n real numbers, $\kappa_1, \kappa_2, \kappa_3, \ldots \kappa_{n-1}, \kappa_n$, whose sum is $\kappa_{1.n}$. The system of n thin doublets of the standard type of powers $\kappa_1, \kappa_2, \kappa_3, \ldots \kappa_{n-1}, \kappa_n$, placed in contact will be achromatic and of power $\kappa_{1.n}$. If the odd numbers have their crown components leading and the even numbers their flint components leading, and the curvature of the last surface of the λth component is equal and opposite to the curvature of the first surface of the $(\lambda+1)$th component for all values of λ from 1 to $n-1$, the resulting system consists of $n+1$ lenses, the odd numbers being of crown glass and the even numbers of flint glass. The powers of these lenses are

$\kappa_1 K, (\kappa_1+\kappa_2)K', (\kappa_2+\kappa_3)K, (\kappa_3+\kappa_4)K', \ldots (\kappa_{n-1}+\kappa_n)K, \kappa_n K'$
if n is odd, and

$\kappa_1 K, (\kappa_1+\kappa_2)K', (\kappa_2+\kappa_3)K, (\kappa_3+\kappa_4)K', \ldots (\kappa_{n-1}+\kappa_n)K', \kappa_n K$
if n is even.

When n thin systems are combined to form a new thin system, the aberration coefficients of the latter may be calculated from those of the components by means of the equations

$$\kappa_{1,n}\varpi_{1.n}=\sum_1^n \kappa_\lambda\varpi_\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\kappa^2_{1.n}(B'_{1.n}-B_{1.n})=\sum_1^n \{\kappa_\lambda^2(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda\}. \quad (2)$$

and

$$\begin{aligned}\kappa^3_{1.n}&(2C_{1.n}+\varpi_{1.n})\\ &=\sum_1^n\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1.\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2(B_\lambda'-B_\lambda)\\ &\qquad+(\kappa_{1.\lambda}-\kappa_{\lambda,n})^2\kappa_\lambda\varpi_\lambda\}. \quad . \quad . \quad (3)\end{aligned}$$

The first equation follows immediately from the definition of ϖ. The other two may be verified by induction. They are known to hold for two thin systems in contact. Assuming that (2) holds for a definite value of n and for $n=2$, let the system 1 to n be taken as the first component in the latter case, and

$(n+1)$ as the second component. Thus, in addition to equation (2) the relation

$$\kappa^2_{1,n+1}(B'_{1,n+1}-B_{1,n+1})$$
$$=\kappa^2_{1,n}(B'_{1,n}-B_{1,n})+\kappa^2_{n+1}(B'_{n+1}-B_{n+1})$$
$$+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}-\kappa_{1,n}\kappa_{n+1}\varpi_{n+1}$$

is given. Substitute in this equation from (2) for $\kappa^2_{1,n}(B'_{1,n}-B_{1,n})$. Then

$$\kappa^2_{1,n+1}(B'_{1,n+1}-B_{1,n+1})$$
$$=\sum_1^n\{\kappa^2_\lambda(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda\}$$
$$+\kappa^2_{n+1}(B'_{n+1}-B_{n+1})+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}-\kappa_{1,n}\kappa_{n+1}\varpi_{n+1}$$
$$=\sum_1^{n+1}\{\kappa_\lambda^2(B'_\lambda-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda\varpi_\lambda\}$$
$$-\kappa_{n+1}\sum_1^n\kappa_\lambda\varpi_\lambda+\kappa_{n+1}\kappa_{1,n}\varpi_{1,n}$$
$$=\sum_1^{n+1}\{\kappa_\lambda^2(B_\lambda'-B_\lambda)-(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda\varpi_\lambda\}$$

by (1). It follows that (2) is always true for any compound thin lens.

Similarly from (3) and from the special case

$$\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})$$
$$=\kappa^3_{1,n}(2C_{1,n}+\varpi_{1,n})+\kappa^3_{n+1}(2C_{n+1}+\varpi_{n+1})$$
$$+2\kappa_{n+1}\kappa^2_{1,n}(B'_{1,n}-B_{1,n})-2\kappa_{1,n}\kappa^2_{n+1}(B'_{n+1}-B_{n+1})$$
$$+\kappa^2_{n+1}\kappa_{1,n}\varpi_{1,n}+\kappa^2_{1,n}\kappa_{n+1}\varpi_{n+1}$$

it follows that

$$\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})$$
$$=\sum_1^{n+1}\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1,\lambda}-\kappa_{\lambda,\ n})\kappa_\lambda^2(B_\lambda'-B_\lambda)$$
$$+(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})^2\kappa_\lambda\varpi_\lambda\}$$
$$-2\kappa_{n+1}\sum_1^n\kappa_\lambda^2(B_\lambda'-B_\lambda)+2\kappa_{n+1}\kappa^2_{1,n}(B'_{1,n}-B_{1,n})$$
$$-\kappa^2_{n+1}\sum_1^n\kappa_\lambda\varpi_\lambda+2\kappa_{n+1}\sum_1^n(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda+\kappa^2_{n+1}\kappa_{1,n}\varpi_{1,n}$$

or by (1) and (2)

$$\kappa^3_{1,n+1}(2C_{1,n+1}+\varpi_{1,n+1})$$
$$=\sum_1^{n+1}\{\kappa_\lambda^3(2C_\lambda+\varpi_\lambda)-2(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})\kappa_\lambda^2(B_\lambda'-B_\lambda)$$
$$+(\kappa_{1,\lambda}-\kappa_{\lambda,n+1})^2\kappa_\lambda\varpi_\lambda\}$$

which is the result required for inductive verification.

Let σ_λ denote the value of $\Sigma\kappa R$ for a lens similar to the λth component, but of unit power. The corresponding quantity for the λth component will be $\kappa_\lambda{}^2\sigma_\lambda$, since each factor will be altered in the ratio $\kappa_\lambda : 1$. For the complete system the sum is therefore given by

$$\kappa^2{}_{1,n}\sigma_{1,n}=\Sigma\kappa_\lambda{}^2\sigma_\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

There are $n-1$ geometrical conditions to be satisfied, if for all values of λ the last surface of the λth component is to be equal in curvature to the first surface of the $(\lambda+1)$th component, so that the two may be cemented together. If the curvature of each surface of the λth component is greater by r_λ than when C_λ is a minimum

$$2(B_\lambda{}'-B_\lambda)-\sigma_\lambda=2\frac{r_\lambda}{\kappa_\lambda}\{2(1+\varpi_\lambda)-1\}=2\frac{r_\lambda}{\kappa_\lambda}(1+2\varpi_\lambda)$$

and therefore from (2) and (4) $C_{1,n}$ will be a minimum if

$$\Sigma\kappa_\lambda r_\lambda(1+2\varpi_\lambda)=\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda\varpi_\lambda. \quad . \quad . \quad . \quad (5)$$

It is convenient to satisfy this condition and to calculate the corresponding values of $C_{1,n}$ and $B'_{1,n}-B_{1,n}$. The calculation of the coefficients for any other conformation of the system may be derived immediately in terms of these quantities, of $\varpi_{1,n}$, and of the change in curvature which will transform the system from the one conformation to the other. When each component is an achromatic combination of the same two glasses the ϖ of every component and of the compound system has the value

$$\frac{K}{\mu}+\frac{K'}{\mu'},$$

where μ and μ' are the refractive indices of the crown and flint glasses. Since

$$\Sigma(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the condition (5) then takes the simple form

$$\Sigma\kappa_\lambda r_\lambda=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

This condition, together with the $n-1$ geometrical conditions already mentioned enables r_λ to be determined uniquely for all values of λ. Equations (2) and (3) may be simplified when the conditions $\varpi_\lambda=\varpi_{1,n}=\varpi$ are always true. In consequence of (6) equation (2) becomes

$$\kappa^2{}_{1,n}(B'_{1,n}-B_{1,n})=\Sigma\kappa_\lambda{}^2(B_\lambda{}'-B_\lambda). \quad . \quad . \quad . \quad (8)$$

Again, $\kappa_{1,\lambda}-\kappa_{\lambda,n}=\kappa_{1,\lambda}+\kappa_{1,\lambda-1}-\kappa_{1,n}$

and $\Sigma\kappa_\lambda(\kappa_{1,\lambda}+\kappa_{1,\lambda-1})=\kappa^2_{1,n}$

and therefore

$$\begin{aligned}\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2 &=\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{1,\lambda-1})^2+4\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}-\kappa^3_{1,n}\\ &=\Sigma\kappa_\lambda^3+4\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}-\kappa^3_{1,n}\end{aligned}$$

but

$$\begin{aligned}\Sigma\kappa^3 &=\Sigma(\kappa_{1,\lambda}-\kappa_{1,\lambda-1})^3\\ &=\kappa^3_{1,n}-3\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1}.\end{aligned}$$

and thus

$$3\Sigma\kappa_\lambda(\kappa_{1,\lambda}-\kappa_{\lambda,n})^2=\kappa^3_{1,n}-\Sigma\kappa_\lambda^3. \quad . \quad . \quad . \quad . \quad (9)$$

Substitute this value in (3). The result is

$$\kappa^3_{1,n}(3C_{1,n}+\varpi) =\sum_1^n \kappa_\lambda^3(3C_\lambda+\varpi)-3(\kappa_{1,\lambda}-\kappa_{\lambda,n})\kappa_\lambda^2(B'_\lambda-B_\lambda) \quad . \quad . \quad (10)$$

It is convenient to refer the components to a standard lens of the same "type." One thin lens may be said to be of the same type as another, when the same glasses are employed in the same order, and the first can be made geometrically similar to the second by the addition of the same curvature to each surface. The standard lens will be assumed to be of unit focal length and will have that form for which C is a minimum. The aberration coefficients of the standard lens will be distinguished by the suffix 0. The curvature of the first surface of the standard lens will be denoted by S, and that of the last surface by S'.

The $n-1$ geometrical conditions which must be satisfied when the components are all of the same type as the standard lens, the odd numbers the same way round as the standard and the even numbers reversed, are

$$\kappa_\lambda S'+r_\lambda=-\kappa_{\lambda+1}S'+r_{\lambda+1},$$

when λ is odd, and

$$-\kappa_\lambda S+r_\lambda=\kappa_{\lambda+1}S+r_{\lambda+1},$$

when λ is even.

These conditions, as well as (7) are satisfied if

$$\left.\begin{aligned}&2r_\lambda=-(\kappa_{1,\lambda}-\kappa_{\lambda+1,n})S+(\kappa_{1,\lambda-1}-\kappa_{\lambda,n})S'+\frac{\theta}{\kappa_{1,n}}(S+S')\\ &\text{when } \lambda \text{ is odd, and}\\ &2r_\lambda=-(\kappa_{1,\lambda-1}-\kappa_{\lambda,n})S+(\kappa_{1,\lambda}-\kappa_{\lambda+1,n})S'+\frac{\theta}{\kappa_{1,n}}(S+S')\end{aligned}\right\} . (11)$$

when λ is even, where

$$\theta=\kappa_1^2-\kappa_2^2+\kappa_3^2-\kappa_4^2+ \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The aberration coefficients of the components are connected with those of the standard lens by the two equations

$$C_\lambda = C_0 + \left(\frac{r_\lambda}{\kappa_\lambda}\right)^2 (1+2\varpi)$$

and $$B'_\lambda - B_\lambda = B_0' - B_0 + 2\frac{r_\lambda}{\kappa_\lambda}(1+\varpi)$$

when λ is odd, and

$$B'_\lambda - B_\lambda = -(B_0' - B_0) + 2\frac{r_\lambda}{\kappa_\lambda}(1+\varpi)$$

when λ is even. . . . (13)

Use large heavy type for the aberration coefficients of the complete lens, and let the suffix 0 be added when equation (7) is satisfied. Substitution from (13) in equation (8) gives

$$\kappa^2_{1,n}(\mathbf{B}_0' - \mathbf{B}_0) = \theta(B_0' - B_0) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Let φ be written for

$$\Sigma(-)^\lambda(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa^2_\lambda . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Then

$$\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda^2(B'_\lambda - B_\lambda)$$
$$= -\varphi(B_0' - B_0) + 2(1+\varpi)\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda r_\lambda,$$

and

$$\Sigma\kappa_\lambda^3(3C_\lambda + \varpi_\lambda) = (3C_0 + \varpi)\Sigma\kappa_\lambda^3 + 3(1+2\varpi)\Sigma\kappa_\lambda r_\lambda^2.$$

Now

$$2\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda r_\lambda$$
$$= (S'-S)\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})^2\kappa_\lambda + (S+S')\Sigma(-)^\lambda(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda^2$$
$$+ (S+S')\frac{\theta}{\kappa_{1,n}}\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda$$
$$= \tfrac{1}{3}(S'-S)(\kappa^3_{1,n} - \Sigma\kappa_\lambda^3) + \varphi(S+S')$$

by (6), (9) and (15).

Also

$$4\Sigma\kappa_\lambda r_\lambda^2$$
$$= (S'-S)^2\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})^2\kappa_\lambda + (S+S')^2\Sigma\kappa_\lambda^3 + (S+S')^2\frac{\theta^2}{\kappa^2_{1,n}}\Sigma\kappa_\lambda$$
$$+ 2(S'^2 - S^2)\Sigma(-)^\lambda(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda^2 - 2(S+S')^2\frac{\theta^2}{\kappa_{1,n}}$$
$$+ 2(S'^2 - S^2)\frac{\theta}{\kappa_{1,n}}\Sigma(\kappa_{1,\lambda} - \kappa_{\lambda,n})\kappa_\lambda$$
$$= \tfrac{1}{3}(S'-S)^2(\kappa^3_{1,n} - \Sigma\kappa_\lambda^3) + (S+S')^2\left(\Sigma\kappa_\lambda^3 - \frac{\theta^2}{\kappa_{1,n}}\right)$$
$$+ 2\varphi(S'^2 - S^2),$$

and, therefore, from (10)

$$\begin{aligned}
&4\kappa^3_{1,n}(3\mathbf{C}_0+\varpi)\\
&=4(3C_0+\varpi)\Sigma\kappa_\lambda{}^3+12\varphi(B_0'-B_0)\\
&\quad+(1+2\varpi)\Big\{(S'-S)^2(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)+3(S+S')^2\Big(\Sigma\kappa_\lambda{}^3-\frac{\theta^2}{\kappa_{1,n}}\Big)\\
&\qquad\qquad+6\varphi(S'^2-S^2)\Big\}\\
&\quad-4(1+\varpi)\{(S'-S)(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)+3\varphi(S+S')\}.
\end{aligned}$$

Let A denote the spherical aberration coefficient for a component having its first surface plane, and similarly A' the reversed spherical aberration coefficient for a component having its last surface plane. These coefficients may be found by impressing the general curvatures $-S$ and $-S'$ on the standard lens. Therefore,

$$\begin{aligned}
A&=A_0+2S(1+\varpi)+S^2(1+2\varpi)\\
&=-(B_0'-B_0)+C_0+1+\varpi+2S(1+\varpi)+S^2(1+2\varpi).
\end{aligned}$$

and

$$\begin{aligned}
A'&=A_0'-2S'(1+\varpi)+S'^2(1+2\varpi)\\
&=B_0'-B_0+C_0+1+\varpi-2S'(1+\varpi)+S'^2(1+2\varpi).
\end{aligned}$$

from which it may be seen that the coefficient of φ in the above expression is $6(A'-A)$.

The coefficient of $\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3$ is

$$\begin{aligned}
&(1+2\varpi)(S'-S)^2-4(1+\varpi)(S'-S)\\
&=-(1+2\varpi)(S'+S)^2+2\{(1+2\varpi)(S'^2+S^2)-2(1+\varpi)(S'-S)\}\\
&=-(1+2\varpi)(S'+S)^2+2\{A+A'-2C_0-2\varpi-2\},
\end{aligned}$$

and the equation may therefore be written

$$\begin{aligned}
4\kappa^3_{1,n}(3\mathbf{C}_0+\varpi)=4(3C_0+\varpi)\Sigma\kappa_\lambda{}^3&\\
+2(A+A'-2C_0-2\varpi-2)(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)&\\
+(S+S')^2(1+2\varpi)\Big(4\Sigma\kappa_\lambda{}^3-\kappa^3_{1,n}-\frac{3\theta^2}{\kappa_{1,n}}\Big)&\\
+6(A'-A)\varphi,&
\end{aligned}$$

or, somewhat more symmetrically

$$\begin{aligned}
&\{4\mathbf{C}_0+2\varpi+1-\tfrac{1}{2}(A+A')\}\kappa^3_{1,n}+(S+S')^2(1+2\varpi)\frac{\theta^2}{\kappa_{1,n}}\\
&=\tfrac{1}{3}\{4C_0+2\varpi+1-\tfrac{1}{2}(A+A')+(S+S')^2(1+2\varpi)\}(4\Sigma\kappa_\lambda{}^3-\kappa^3_{1,n})\\
&\qquad+2(A'-A)\varphi. \quad\ldots\ldots\ldots \quad (16)
\end{aligned}$$

Equations (14) and (16) express the aberration coefficients of the compound lens in terms of those of the standard lens. The curvatures of the external surfaces of the compound lens.

are $\kappa_1 S+r_1$ and $\kappa_n S'+r_n$ if n is odd, or $-\kappa_n S+r_n$ if n is even, *i.e.*, the curvature of the first surface is

$$\tfrac{1}{2}\left\{\kappa_{1,n}(S-S')+\frac{\theta}{\kappa_{1,n}}(S+S')\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and that of the last surface is

$$\tfrac{1}{2}\left\{-\kappa_{1,n}(S-S')+\frac{\theta}{\kappa_{1,n}}(S+S')\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The curvatures of intermediate surfaces are most simply obtained from these by noting that the successive surfaces differ from one another in curvature by the amounts

$$\kappa_1 R,\ (\kappa_1+\kappa_2)R',\ (\kappa_2+\kappa_3)R,\ (\kappa_3+\kappa_4)R' . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Application to the Calculation of Objectives.

The two aberration conditions which an achromatic thin lens can satisfy may always be expressed in the form that $\mathbf{C}$ and $\mathbf{B}'-\mathbf{B}$ are to have definite values. The most frequent conditions are the absence of first order spherical aberration and coma for a definite magnification. Denote this magnification by m. The conditions to be satisfied are

$$4\mathbf{C}+2\varpi+1=\left(\frac{1+m}{1-m}\right)^2(5+2\varpi),$$

and

$$\mathbf{B}'-\mathbf{B}=\frac{1+m}{1-m}(2+\varpi).$$

These values are to be derived by bending the lens from the form in which $\mathbf{C}$ is a minimum, and require the relation

$$\left\{\frac{1+m}{1-m}+(\mathbf{B}'_0-\mathbf{B}_0)(1+2\varpi)(2+\varpi)\right\}^2$$
$$=(1+\varpi)^2\{(\mathbf{B}'_0-\mathbf{B}_0)^2(5+2\varpi)(1+2\varpi)+4\mathbf{C}_0+2\varpi+1\}$$

between the aberration coefficients of the unbent lens to be satisfied. This condition implies that a relation must be satisfied by the κ's, and this relation is evidently of the fourth order. It will be shown that it can always be reduced to a linear condition connecting two κ's, and that the determination of a lens satisfying the two aberration conditions requires the solution of a quadratic equation. This might be inferred from the case of triple objectives, since a triple objective may be made to add any required amount to the aberrations arising in the rest of the system.

The simplest method of solution is to assume definite values for $\kappa_{1,n}$ and for θ. The curvature r must then be added to each of the curvatures given by equations (17), (18) and (19), where

$$\mathrm{B}' - \mathrm{B} = \mathrm{B}_0' - \mathrm{B}_0 + 2\frac{r}{\kappa_{1,n}}(1+\varpi).$$

It is evident from this equation that all lenses satisfying the same conditions have their r and θ connected by a linear equation. This is equivalent to saying that the r and θ are determined by the curvatures of the external surfaces of the system, and these may, if preferred, be taken as the independent variables. To complete the solution it is necessary to choose the κ's so that the value of C_0 given by equation (16) agrees with that which makes

$$\mathrm{C} = \mathrm{C}_0 + \left(\frac{r}{\kappa_{1,n}}\right)^2(1+2\varpi).$$

It is obvious from the equations which have been found that if θ, r are values which satisfy the conditions when a crown component leads, the corresponding values with a flint component leading, and unaltered curvatures for the first and last surfaces, are $-\theta$, r.

Quadruple Objectives.

Take first a system consisting of two crown and two flint lenses. The identities

$$\theta = \kappa_1^2 - \kappa_2^2 + \kappa_3^2 = \kappa^2_{1,3} - 2(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)$$

$$\kappa_1^3 + \kappa_2^3 + \kappa_3^3 = \kappa^3_{1,3} - 3(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_1)$$

and

$$\varphi = \kappa_1^2(\kappa_2+\kappa_3) + \kappa_2^2(\kappa_1-\kappa_3) - \kappa_3^2(\kappa_1+\kappa_2)$$

$$= (\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_1-\kappa_3),$$

show that if definite values are assigned to $\kappa_{1,3}$ and θ, equation (16) becomes a linear relation between κ_1 and κ_3. From this relation and from the assumed value of $\kappa_{1,3}$ the powers of two of the component doublets may be expressed as linear functions of the power of the remaining component. There are therefore two solutions when $\kappa_{1,3}$ and θ are given. There will be two solutions to the corresponding problem when a flint lens leads instead of a crown. With two given glasses there are therefore four infinite series of objectives possible.

The four triple objectives will occur as the special solutions $\kappa_1=0$ and $\kappa_3=0$ in both crown leading and flint leading solutions. $\kappa_2=0$ is, of course, not a triple objective.

The linear relation between κ_1 and κ_3 in this case may be written

$$\{4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A'\}\kappa_1$$
$$+\{4C_0+2\varpi+1+(S+S')^2(1+2\varpi)-A\}\kappa_3$$
$$=\frac{(C_0-\mathbf{C}_0)\kappa^3_{1,3}}{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)}+(S+S')^2(1+2\varpi)\left\{\kappa_{1,3}-\frac{(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)}{\kappa_{1,3}}\right\}$$

It is evident that $(S+S')^2$ only disappears from the equation if κ_1 or κ_3 or $\kappa_1+\kappa_2$ or $\kappa_2+\kappa_3$ vanishes. In the first case the equation becomes

$$(4\mathbf{C}_0+2\varpi+1-A)\kappa^2_{2,3}=(4C_0+2\varpi+1-A)(\kappa_2-\kappa_3)^2,$$

and in the second

$$(4\mathbf{C}_0+2\varpi+1-A')\kappa^2_{1,2}=(4C_0+2\varpi+1-A')(\kappa_1-\kappa_2)^2,$$

the equations previously found for triple objectives.

In the last two cases the system reduces to a doublet and the equation becomes $\mathbf{C}_0=C_0$.

Quintuple Objectives.

The solution of quintuple objectives may be carried out in a very similar way to that used in the previous case. If $\kappa_3+\kappa_4=0$, $\kappa_{1,4}$, θ, and $\Sigma\kappa_\lambda^3$ reduce to $\kappa_1+\kappa_2$, $\kappa_1^2-\kappa_2^2$, and $\kappa_1^3+\kappa_2^3$ respectively, and therefore

$$\kappa^4_{1,4}+3\theta^2-4\kappa_{1,4}\Sigma\kappa_\lambda^3$$

vanishes. Thus $\kappa_3+\kappa_4$ is a factor of this expression, and similarly $\kappa_1+\kappa_2$, $\kappa_2+\kappa_3$, and $\kappa_1+\kappa_4$ must be factors. When $\kappa_1=\kappa_2=\kappa_3=\kappa_4=1$, the value of the expression is 3×4^3, and therefore

$$\kappa^4_{1,4}+3\theta^2-4\kappa_{1,4}\Sigma\kappa_\lambda^3=12(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)(\kappa_1+\kappa_4).$$

Again for a triplet $\phi=\kappa_1\kappa_2(\kappa_1+\kappa_2)$, and the present system reduces to a triplet if $\kappa_1+\kappa_2$, $\kappa_2+\kappa_3$, or $\kappa_3+\kappa_4$ vanishes. Therefore these are all factors of

$$\kappa^3_{1,4}-\Sigma\kappa_\lambda^3-3\phi.$$

When each κ is equal to unity the value of this expression is 3×4^2, and thus

$$\kappa^3_{1,4}-\Sigma\kappa_\lambda^3-3\phi=6(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4).$$

If then definite values are assigned to $\kappa_{1,4}$, θ and either $\Sigma\kappa_\lambda^2$ or φ, the condition (16) becomes a simple equation for $\kappa_1-\kappa_4$.

From the values of $\kappa_1+\kappa_4$, $\kappa_{1,4}$, and θ, any three of the four quantities κ_1, κ_2, κ_3, κ_4 are found as linear functions of the fourth, and either of the two identities given above becomes a quadratic equation for the determination of this κ. There are thus the same number of series of objectives as before, but each series is now doubly infinite.

The equation for $\kappa_1+\kappa_4$ when $\kappa_{1,4}$, θ and $(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)$ are taken as independent variables is

$$\begin{aligned}(4C_0+2\varpi+1-A')\kappa^4_{1,4}-(4C_0+2\varpi+1-A')\theta^2 \\ +4(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4)[\{4C_0+2\varpi+1 \\ +(S+S')^2(1+2\varpi)-A'\}(\kappa_1+\kappa_4)+(A'-A)\kappa_{1,4}]=0.\end{aligned}$$

Multiple Objectives.

The results obtained in the case of objectives of four and five lenses may be shown to hold generally. Put

$$\begin{aligned}\psi=&(\kappa_1+\kappa_2)(\kappa_2+\kappa_3)(\kappa_3+\kappa_4+ \quad \dots\dots \quad +\kappa_n) \\ &+(\kappa_1+\kappa_2+\kappa_3+\kappa_4)(\kappa_4+\kappa_5)(\kappa_5+\kappa_6+ \quad \dots \quad +\kappa_n) \\ &+(\kappa_1+\kappa_2+ \dots +\kappa_5+\kappa_6)(\kappa_6+\kappa_7)(\kappa_7+\kappa_8+ \quad \dots \quad +\kappa_n) \\ &+ \dots\end{aligned}$$

and

$$\begin{aligned}\psi'=&\kappa_1(\kappa_1+\kappa_2)(\kappa_2+\kappa_3+\kappa_4+ \quad \dots\dots \quad +\kappa_n) \\ &+(\kappa_1+\kappa_2+\kappa_3)(\kappa_3+\kappa_4)(\kappa_4+\kappa_5+ \quad \dots \quad +\kappa_n) \\ &+(\kappa_1+\kappa_2+\kappa_3+\kappa_4+\kappa_5)(\kappa_5+\kappa_6)(\kappa_6+\kappa_7+ \quad \dots \quad +\kappa_n) \\ &+ \dots ,\end{aligned}$$

the general term in each case being

$$\kappa_{1,\lambda}(\kappa_\lambda+\kappa_{\lambda+1})\kappa_{\lambda+1,n}$$

where in ψ, λ is given all even values less than n, and in ψ' all odd values below the same limit. Then

$$\begin{aligned}\psi+\psi'&=\sum_1^n\kappa_\lambda(\kappa_{1,\lambda}\kappa_{\lambda+1,n}+\kappa_{1,\lambda-1}\kappa_{\lambda,n}) \\ &=\kappa_{1,n}\Sigma\kappa_\lambda(\kappa_{1,\lambda}+\kappa_{1,\lambda-1})-\Sigma\kappa_\lambda(\kappa^2_{1,\lambda}+\kappa^2_{1,\lambda-1}) \\ &=\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3-2\Sigma\kappa_\lambda\kappa_{1,\lambda}\kappa_{1,\lambda-1} \\ &=\tfrac{1}{3}(\kappa^3_{1,n}-\Sigma\kappa_\lambda{}^3)\end{aligned}$$

by a result already proved, and

$$\begin{aligned}\psi'-\psi &= -\Sigma(-)^{\lambda}\kappa_{\lambda}\{\kappa_{1,\lambda}\kappa_{\lambda+1,n}-\kappa_{1,\lambda-1}\kappa_{\lambda,n}\}\\ &= -\Sigma(-)^{\lambda}\kappa_{\lambda}\{\kappa_{1,\lambda}(\kappa_{\lambda,n}-\kappa_{\lambda})-(\kappa_{1,\lambda}-\kappa_{\lambda})\kappa_{\lambda,n}\}\\ &= \Sigma(-)^{\lambda}\kappa_{\lambda}{}^{2}(\kappa_{1,\lambda}-\kappa_{\lambda,n})\\ &= 0.\end{aligned}$$

Substitute for φ and $\Sigma\kappa_{\lambda}{}^{3}$ in (16) from these expresisons. This equation may then be written,

$$\begin{aligned}&4(\mathrm{C}_0-C_0)\kappa^4{}_{1,n}+(S+S')^2(1+2\varpi)(\theta^2-\kappa^4{}_{1,n})\\ &+4(4C'_0+2\varpi+1+(S+S')^2(1+2\varpi)-A)\psi\kappa_{1,n}\\ &+4(4C'_0+2\varpi+1+(S+S')^2(1+2\varpi)-A')\psi'\kappa_{1,n}\\ &\qquad=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)\end{aligned}$$

Now, assume that $\kappa_{1,n}$, θ, and all but three consecutive κ's have been fixed arbitrarily, or what comes to the same thing, that the curvatures of all but four successive surfaces are given. Let the unknown κ's be $\kappa_{\lambda-1}$, κ_{λ}, and $\kappa_{\lambda+1}$. From $\kappa_{1,n}$ the value of $\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1}$ is known, and from θ that of $\kappa^2{}_{\lambda-1}-\kappa^2{}_{\lambda}+\kappa^2{}_{\lambda+1}$ is known. Since the last is equal to

$$(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})^2-2(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$$

it follows that $(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$ is known. The unknown contributions to ψ and ψ' are in the one case

$$\begin{aligned}&(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda-2})(\kappa_{\lambda-2}+\kappa_{\lambda-1})(\kappa_{\lambda-1}+\kappa^{\lambda}+\ \ldots\ +\kappa_n)\\ &+(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})(\kappa_{\lambda+1}+\kappa_{\lambda+2}+\ \ldots\ +\kappa_n),\end{aligned}$$

and in the other

$$\begin{aligned}&(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda-1})(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1}+\ \ldots\ +\kappa_n)\\ &\quad+(\kappa_1+\kappa_2+\ldots+\kappa_{\lambda+1})(\kappa_{\lambda+1}+\kappa_{\lambda+2})(\kappa_{\lambda+2}+\kappa_{\lambda+3}+\ldots+\kappa_n).\end{aligned}$$

In the first and last of these four lines the first and last factors are known in each case, so that the first line is of the form $a\kappa_{\lambda-1}+b$, and the last of the form $c\kappa_{\lambda+1}+d$, where a, b, c, d are known. Since $(\kappa_{\lambda-1}+\kappa_{\lambda})(\kappa_{\lambda}+\kappa_{\lambda+1})$ is known the second line may be written

$$e+f(\kappa_{\lambda}+\kappa_{\lambda+1})+g\kappa_{\lambda+1}+h(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1},$$

or, since $\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1}$ is known, it takes the form

$$e'+f'\kappa_{\lambda-1}+g\kappa_{\lambda+1}+h(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1}.$$

Now

$$\begin{aligned}\kappa^2{}_{\lambda-1}-\kappa_{\lambda}{}^2+\kappa^2{}_{\lambda+1}+(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})^2&\\ =2\kappa_{\lambda-1}(\kappa_{\lambda-1}+\kappa_{\lambda}+\kappa_{\lambda+1})+2(\kappa_{\lambda}+\kappa_{\lambda+1})\kappa_{\lambda+1},&\end{aligned}$$

and the second line is thus of the form

$$e''+f''\kappa_{\lambda-1}+g''\kappa_{\lambda+1}.$$

The third line may be shown to be of a similar form, and, therefore, both ψ and ψ' are of the form

$$E+F\kappa_{\lambda-1}+G\kappa_{\lambda+1},$$

that is to say, equation (16) or (20) is a linear relation between $\kappa_{\lambda-1}$ and $\kappa_{\lambda+1}$. It follows that the solution is completed by finding the roots of a quadratic equation as in previous cases.

The formula (20) is easily remembered if it is noted that ψ multiplies the term containing A, the spherical aberration of a doublet with a flat crown surface leading, and consists of a series of terms each of which is the sum of the κ's which determine the power of a crown lens in the compound system, multiplied by the sum of all κ's relating to preceding lenses, and by the sum of all κ's relating to succeeding lenses. The other term is determined in a similar way by substituting "flint" for "crown."

Calculation of the Fundamental Constants.

It will usually happen that the glasses from which multiple objectives are made will be restricted to a very small number of varieties, the same two glasses being frequently used for many different purposes. The foregoing analysis shows that very few numbers are required in the calculation of any multiple objectives from a cemented doublet, and it is convenient to keep these in a form suitable for immediate reference. The author has found it best to enter these quantities on a card of the stock size used in card-filing cabinets. The amount of space on these is very limited, and it is consequently best to pay some regard to the length of the expressions to be tabulated in deciding upon the most convenient arrangement of the card. The order adopted is in consequence slightly different from the natural order in which the values are obtained.

The arrangement adopted is shown below. The top line defines the glasses to which the card relates, and also contains a reference to the detailed calculations. It is sometimes more convenient to enter log ν/ν' rather than ν/ν', and space has been left for the insertion of log in such cases. The two quantities first determined are K and K', and these are entered in the first column. The next quantities R and R' are most conveniently placed at the bottom of the second column. The absolute first order coefficients for the doublets occupy the second line, and the curvatures of this standard lens complete the first column. The remaining space in the second column

$\mu =$	$\mu' =$	$\nu/\nu' =$	*Ref.*
$K =$	$\varpi =$	$4C_0+2\varpi+1=$	$B'_0-B_0=$
$K' =$	$A =$	$4C_0+2\varpi+1-A=$	$\frac{B'_0-B_0}{2(1+\varpi)}=$
$R_1 =$	$A' =$	$4C_0+2\varpi+1-A'=$	$\frac{2+\varpi}{2(1+\varpi)}=$
$R_2 =$	$R =$	$(R_1+R_3)^2(1+2\varpi)=$	$5+2\varpi=$
			$4(1+2\varpi)=$
$R_3 =$	$R' =$	$\frac{A-A'-B+B'}{2(1+\varpi)}=$	$\frac{R_1-R_3}{2}=$
$4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)-A=$		$4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)-A'=$	

is taken up by A and A', which may be calculated from the formulæ given below, and checked by derivation from the coefficients for the standard lens. These are conveniently placed to enable the quantities required in the calculation of triple objectives to be written down in the middle of the third column. The additional term required for more complex objectives is entered under this, and the two long expressions obtained by adding this to the terms above are entered below the regular columns. The quantities in the last column assist in the determination of the curvatures and in finding the correct value for $\mathbf{C}_0$. There is further space left on the card for the entry of any other numbers which may be frequently required in connection with the two glasses concerned. R_1 and R_3 have been substituted for S and S'.

A may be calculated from any of the formulæ

$$A=(R+R'+1)^2+\frac{\mu'-\mu}{\mu'^2}R\{(R+R'+1)^2-(R+K)^2\}$$

$$=(R+R'+1)^2+\frac{\mu'-\mu}{\mu'}RR'\{2R+R'+1+K\}$$

$$=(R+R'+1)^2(2\frac{K'}{\mu'}+1)-R'\{(1+K')(R+R'+1)+(K+R)\}$$

with similar expressions for A'. The results may be checked from the values of C_0 and B'_0-B_0, or from one of the relations

$$\begin{aligned}4C_0+2\varpi+1+(R_1+R_3)^2(1+2\varpi)&\\ =A+A'-(R+R'+1)^2+(R-R')(K'R-KR')&\\ =A+A'-(2R+1)(2R'+1)-(R-R')(KR-K'R')&\\ =2(A+A')-(R+R'+1)^2(1+2\varpi)-2(R+R'+1).&\end{aligned}$$

Chromatic Differences of First Order Aberrations.

Equations (14) and (20) are expressions for the values of $\mathbf{B}'_0-\mathbf{B}_0$ and $\mathbf{C}_0$, the fundamental aberration coefficients of the thin compound lens in terms of B'_0-B_0 and C_0, the corresponding coefficients of the standard component. The quantities which occur in these equations, $\kappa_{1\ldots n}$, θ, φ, and φ' are simple ratios, and depend only on the way in which the compound lens is built up from its components. They have no connection with the aberrations of the components or with the wave length which is being considered. When a compound lens has been calculated to satisfy given conditions for light of a definite wave-length, the values of $\mathbf{B}'_0-\mathbf{B}_0$ and $\mathbf{C}_0$ for the

same compound lens, but for another wave-length are found, using the same values for $\kappa_{1,n}$, θ, ψ and ψ', and new values for B'_0-B_0, C_0, ϖ, S and S'. The value of r to be used in calculating $\mathrm{B}'-\mathrm{B}$ and C from $\mathrm{B}'_0-\mathrm{B}_0$ and C_0 will, as a rule, vary with the colour. If the power of the component for the new colour is k when it is unity for the first colour, and the quantities relating to the new colour are distinguished by a bar above the letter, equations (17) and (18) show that

$$k\left\{\bar{r}+\frac{\theta}{2\kappa_{1,n}}(\bar{S}+\bar{S}')\right\}=r+\frac{\theta}{2\kappa_{1,n}}(S+S'),$$

and $$k(\bar{S}-\bar{S}')=S-S'.$$

The aberration coefficients for the new colour are given by

$$\bar{\mathrm{B}}'-\bar{\mathrm{B}}=\bar{\mathrm{B}}'_0-\bar{\mathrm{B}}_0+2\frac{\bar{r}}{\kappa_{1,n}}(1+\bar{\varpi}),$$

and $$\bar{\mathrm{C}}=\bar{\mathrm{C}}_0+\left(\frac{\bar{r}}{\kappa_{1,n}}\right)^2(1+2\bar{\varpi}).$$

Evidently all lenses having the same values for the ratios $\kappa^3_{1,n}:\theta\kappa_{1,n}:\psi:\psi'$ will have the same chromatic differences of first order aberrations if made from the same kinds of glass.

Illustrative Quadruple Lenses.

By way of examples a number of quadruple lenses have been calculated. The ratio taken for the power of the crown glass to that of the flint is $10^{0\cdot20}$ and the refractive index of the flint for the standard line is 1·62. In one series the refractive index for the crown is 1·50, and in another series 1·55. Fig. 1 illustrates the composition of the lenses for various values of r when the former crown glass is taken and a crown lens leads. The most interesting branches of the curves, those which contain the triple objectives, are closed. On one side there are infinite branches to the curves, but as the systems to which these correspond involve the use of more powerful elements than a cemented doublet of equal focal length they are not likely to be adopted in practice. The closed curves and the infinite branches are separated by a belt limited by two values of r between which there is no real solution. For values of r greater than the extreme point on the closed curve there is no real solution. It is obvious that any branch of κ_1 or of κ_3 must lie wholly on one side of $\kappa=1$, since this value of κ corresponds to a doublet. As the refractive indices are changed

towards a combination in which a doublet satisfies the first order conditions, the curves for κ_1 and κ_3 will become more pointed, and when the doublet form is reached it appears probable that the closed and infinite branches of the curve concerned will meet in a node on the line $\kappa=1$. A slight alteration

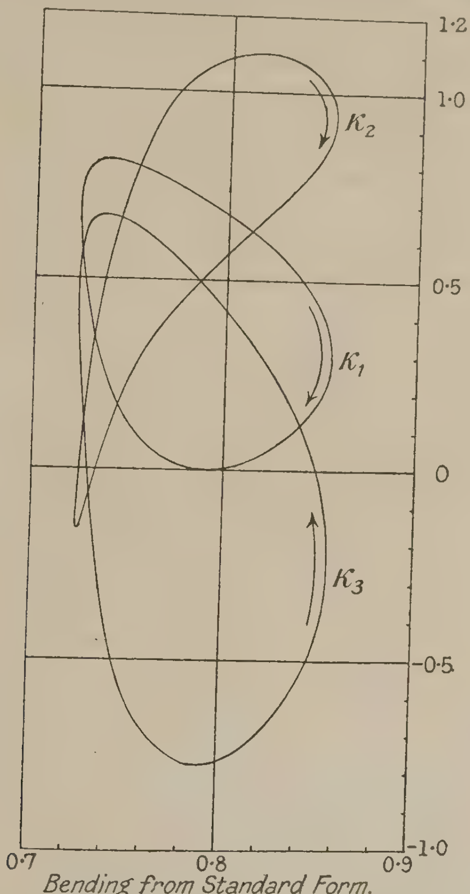


FIG. 1.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING IN TERMS OF DOUBLETS.

Refractive indices 1·50 and 1·62. Log $\nu/\nu'=0{\cdot}20$.

in one of the glasses in either direction will suffice for the connection between the branches to be broken.

It will be noted that while κ_3 crosses the line $\kappa=0$ at widely separated points, κ_1 is almost tangential to this line. This

particular combination of glasses is thus near the limit at which two of the triple solutions become imaginary.

Fig. 2 shows the curvatures of the five surfaces of the quadruple systems. It has already been noted that the external curvatures are linear functions of r, and are thus represented by double straight lines. In the region where the κ's form closed curves, the curvatures of the inner surfaces are also of necessity represented by closed curves. Arrows have been inserted

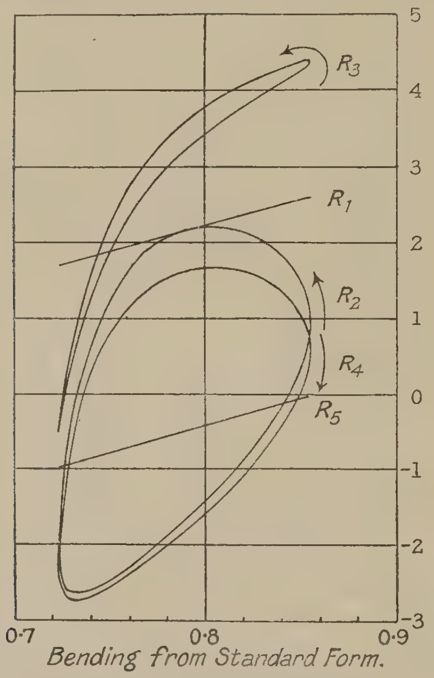


FIG. 2.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.

Refractive indices 1·50 and 1·62. Log $\nu/\nu' = 0{\cdot}20$.

in Figs. 1 and 2 to denote the side of the closed curves which belong to the same series of objectives. The triple objectives in Fig. 2 correspond to the points where R_1 meets R_2 and R_5 meets R_4. Apart from the triple objectives there are 28 objectives which require for their production less than five different finite curvatures. The curves for R_2, R_3 and R_4 all cross the zero line, giving six objectives with flat surfaces. R_5 only just fails to reach this line, but can be made to cross it by a

change in the glasses employed. There are also two points of intersection of R_1 and R_3, two of R_2 and R_4, and two of R_2 and R_5. For somewhat different glasses there will be further intersections between R_1 and R_4, and R_3 and R_5. If the diagram is folded about the line $R=0$ further intersections, corresponding to curvatures of equal amounts but opposite signs, will be

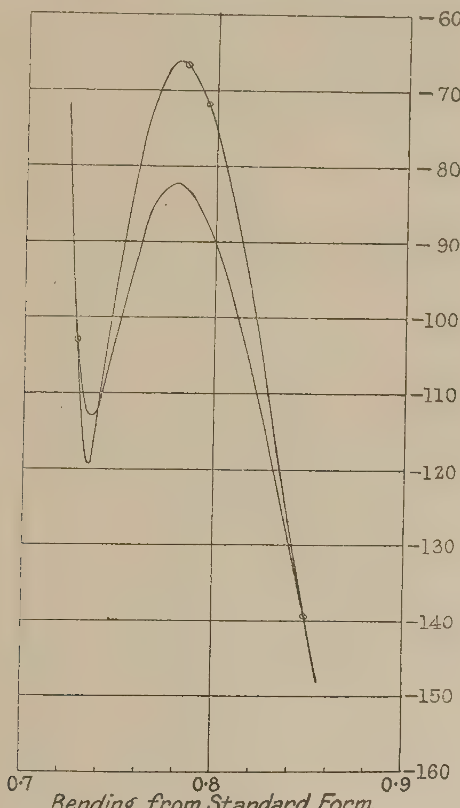


FIG. 3.—SECOND ORDER SPHERICAL ABERRATION FOR OBJECTIVES WITH CROWN LENS LEADING.

obtained. Inspection of the figure shows that R_1 and R_5 will not intersect, and that no further intersections between R_1 and R_3 will be obtained. With these exceptions there will be two new intersections between every pair of curves, giving 16 objectives having two curvatures of equal amount, but of opposite sign. These forms are to be preferred if they are not

appreciably worse than others as regards residual aberrations. A few particularly simple systems are almost attained by the

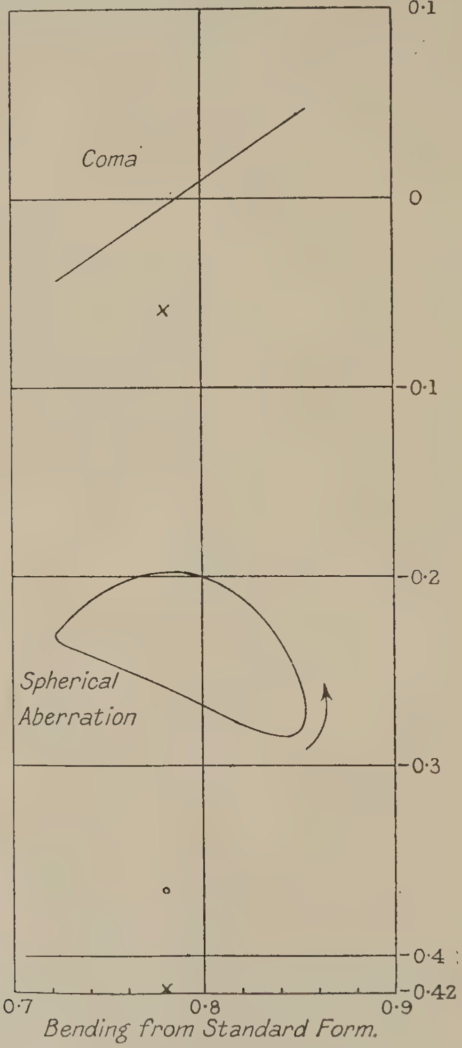


FIG. 4.—CHROMATIC DIFFERENCES OF FIRST ORDER SPHERICAL ABERRATION AND COMA FOR QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.

use of these glasses. For instance, $R_3=0$ in combination with $R_2=R_5$; also $R_1=R_3$ with $R_4=-R_5$. A particularly favour-

able form with different glasses would be $R_2 = R_4$ and $R_3 = R_5$. This is the case requiring minimum curvature on the steepest surfaces; the two flint lenses are exactly alike, and the two crown lenses have one curvature alike. It is also one

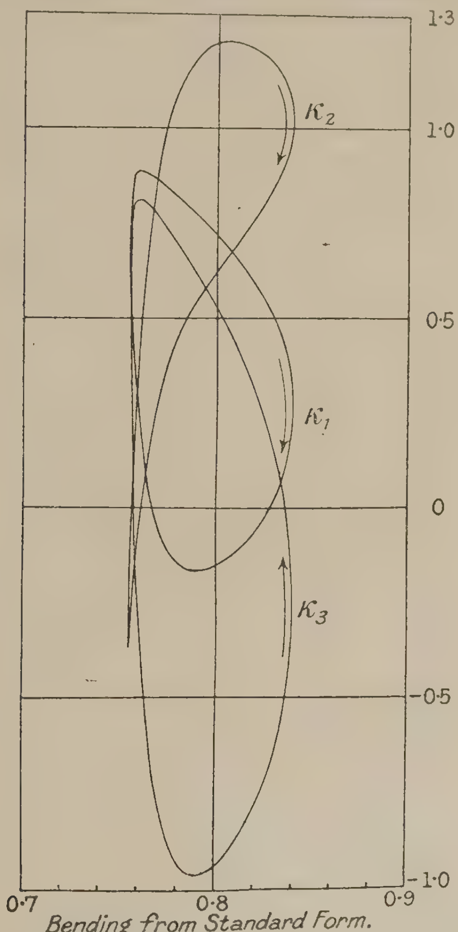


FIG. 5.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING IN TERMS OF DOUBLETS.
Refractive indices 1·55 and 1·62. Log $\nu/\nu' = 0·20$.

of the most favourable forms as regards second order spherical aberration.

Figure 3 shows the second order spherical aberration for the lenses to which Figures 1 and 2 relate. The triple objective

points are distinguished by a circle. It will be observed that the two branches cross one another three times, and that one of the triple objectives is very nearly as good as the best corrected system. All the objectives are over-corrected. Those having the greatest curvatures are decidedly the worst, but the variation of the second order aberration is by no means indicated by the magnitude of the curvatures, the best corrected lenses having curvatures on the large side of the

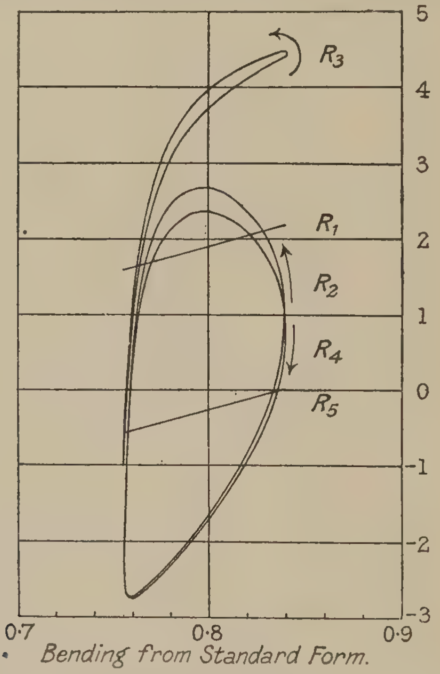


FIG. 6.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH CROWN LENS LEADING.

Refractive indices 1·55 and 1·62. Log $\nu/\nu'=0{\cdot}20$.

mean. Apart from its want of simplicity the most notable character of this curve is the rapid improvement in the neighbourhood of the minimum curvature, where the curves for R_2 and R_4 cross one another.

Figure 4 gives the amounts of first order spherical aberration and coma for light of a different colour. In calculating these curves the refractive indices are taken as 1·50835 and 1·63641, these figures giving the same paraxial focus as 1·50 and 1·62.

The curve for the coma, which depends only on the external curvatures, is a double straight line. The arrow shows the direction in which the spherical aberration curve is described.

Figures 5 and 6 show how 1 and 2 are changed when the refractive index of the crown glass is increased to 1·55, corres-

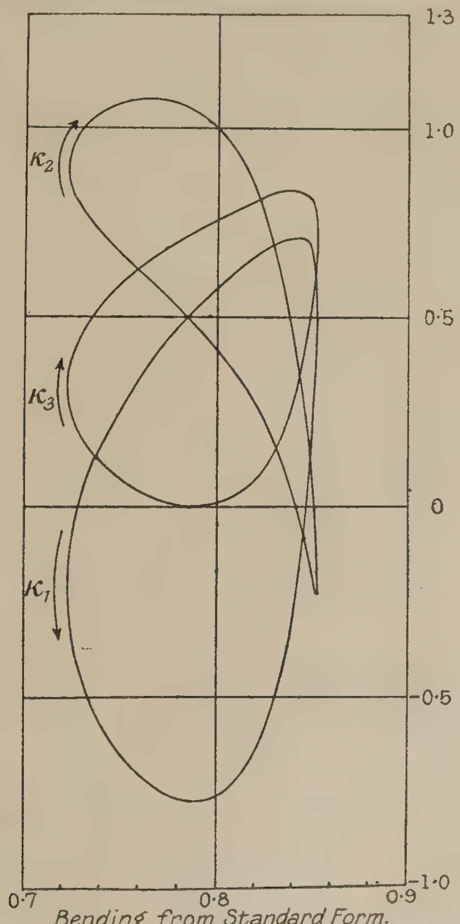


FIG. 7.—COMPOSITION OF QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING IN TERMS OF DOUBLETS.
Refractive indices 1·50 and 1·62. Log $\nu/\nu'=0{\cdot}20$.

ponding approximately to the substitution of a medium barium crown glass for a borosilicate crown. The closed curves extend over a much smaller range of values of r, but

the curvatures of the various surfaces lie within almost unchanged limits.

All the foregoing diagrams relate to quadruple objectives with a crown component leading. Figures 7, 8, 9 and 10 correspond to 1, 2, 3 and 4, but with a flint lens in front. The two sets of curves necessarily intersect in the positions where the quadruple objective degenerates into the triple form. Figure 8 is very similar to Figure 2 as regards the closed curves if these be assumed to be lifted from the paper,

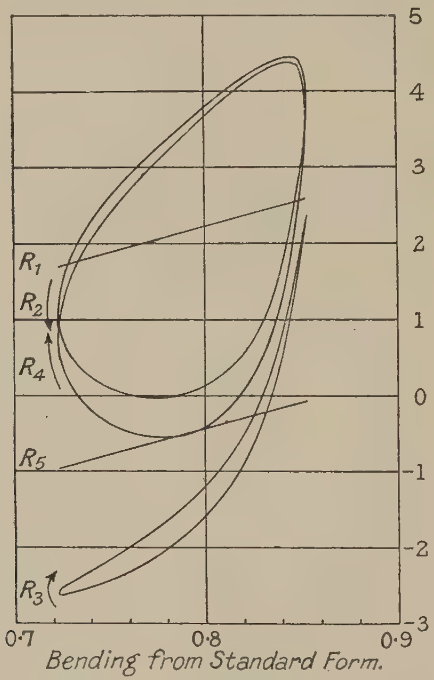


FIG. 8.—CURVATURES OF THE SURFACES OF QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING.

Refractive indices 1·50 and 1·62. Log $\nu/\nu'=0{\cdot}20$.

revolved through two right angles, and replaced with the double straight lines on the same lines as before. It follows that the least curvatures are to be found in the form with a crown lens leading.

For the purpose of comparison with these quadruple objectives, two others have been calculated. The first is the usual form of cemented doublet free from first order spherical

aberration but not free from coma. The second is the usual form of astronomical objective with internal surfaces of different curvatures, but satisfying the conditions for freedom from both spherical aberration and coma. These two objectives are made from the same glasses as the quadruple lenses. The chromatic difference of spherical aberration of the former

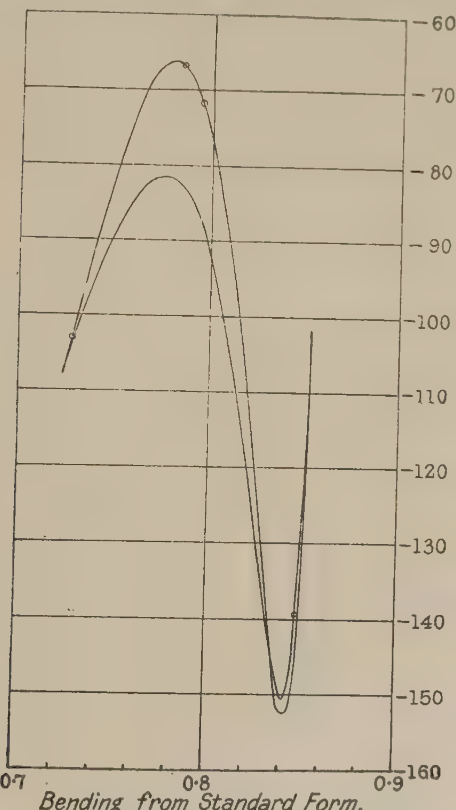


FIG. 9.—SECOND ORDER SPHERICAL ABERRATION FOR OBJECTIVES WITH FLINT LENS LEADING.

is indicated on Fig. 4 by a small circle, and the chromatic differences of spherical aberration and coma of the latter by small crosses. The best forms of triple and quadruple objectives are evidently about twice as good as either the cemented doublet or the astronomical objective as regards the chromatic difference of spherical aberration. For further comparison.

including aberrations of order greater than the second, rays have been traced through zones of these objectives, and through the best corrected triple objectives, and also through the quadruple of minimum curvature. The results of these calculations, for which the author is indebted to Miss Dale, Miss Everett and Mr. Trump, are given in Figs. 11 to 15. The

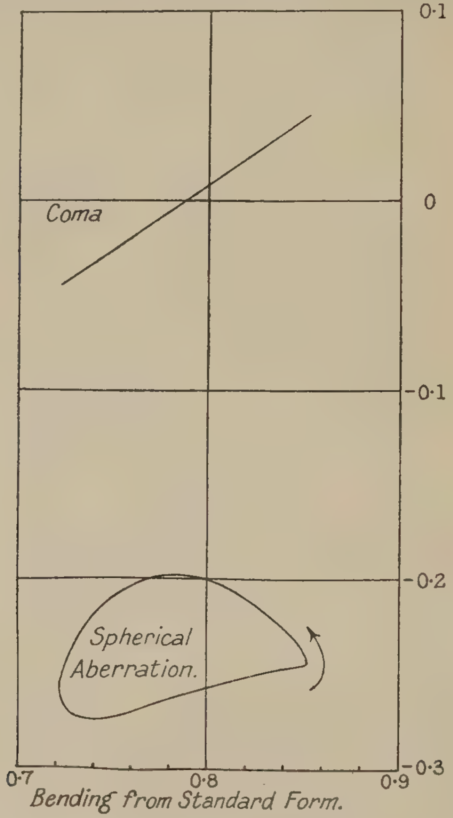


FIG. 10.—CHROMATIC DIFFERENCES OF SPHERICAL ABERRATION AND COMA FOR QUADRUPLE OBJECTIVES WITH FLINT LENS LEADING.

curved line shows the second order spherical aberration for various incident heights. The exact values found from tracing through a number of rays are indicated by small crosses. The dots give the corresponding quantities when the refractive indices bear the values used for calculating the chromatic differences in the aberrations. Thus the separation between

the cross and the curve shows the amount of aberration of orders higher than the second, and the distance between the cross and the dot gives the chromatic difference of aberration. The outstanding features are the relatively bad performance of the astronomical objective and the very small higher order aberrations of the objectives which have small second order aberrations. The conditions satisfied by these lenses are not those which would be selected in use, inasmuch as a compromise

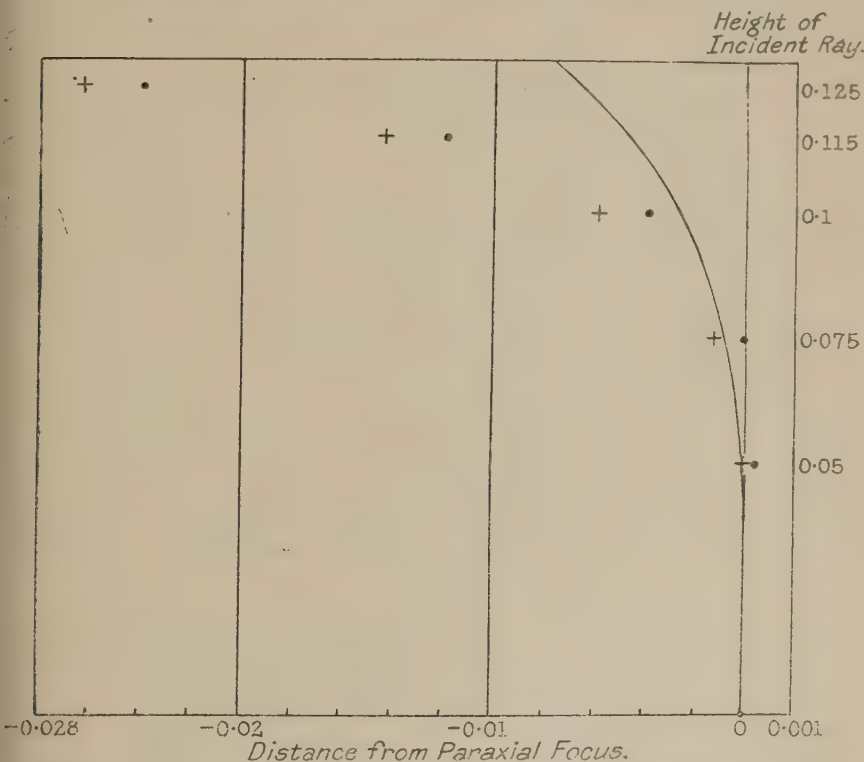


FIG. 11.—SPHERICAL ABERRATION OF UNCEMENTED TELESCOPE OBJECTIVE, USUAL ASTRONOMICAL FORM.

between the aberrations of different orders would be preferred. The first order aberrations are entirely removed in each case, and the outstanding errors shown in the diagrams enable the merits of the different forms for this particular pair of glasses —which it will be realised are quite arbitrary and not necessarily realisable—to be compared. The absolute amount of

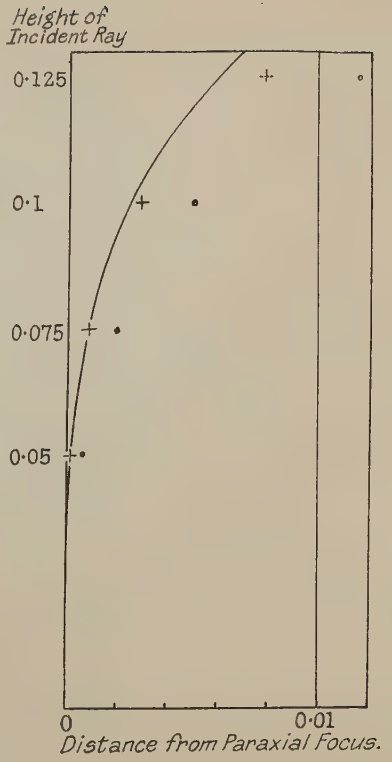


FIG. 12.—SPHERICAL ABERRATION OF CEMENTED DOUBLET WITH CROWN LENS LEADING, FREE FROM FIRST ORDER ABERRATION.

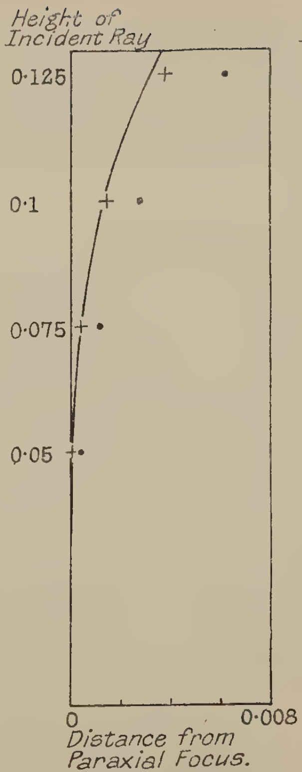


FIG. 13.—SPHERICAL ABERRATION OF TRIPLE OBJECTIVE WITH EXTERNAL CROWN LENSES.

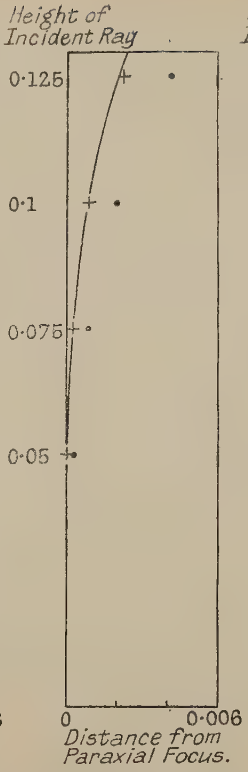


FIG. 14.—SPHERICAL ABERRATION OF TRIPLE OBJECTIVE WITH EXTERNAL FLINT LENSES.

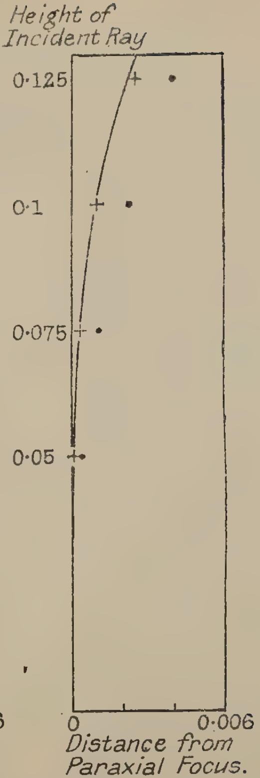


FIG. 15.—SPHERICAL ABERRATION OF QUADRUPLE OBJECTIVE WITH SMALLEST CURVATURES.

spherical aberration for a given colour can be reduced to nearly one eighth of the value shown in these figures for a given maximum aperture by the introduction of a suitable small amount of first order aberration. It would be necessary for such a comparison to be extended over the whole range of possible glasses before the advantages obtainable from the use of four elements in an objective of two glasses could be fully estimated. The calculations described in the present paper show that this class of objective merits more detailed numerical investigation.

ABSTRACT.

The objectives dealt with are cemented combinations of several thin lenses. Two kinds of glass only are employed, the odd elements being of one kind, say, crown, and the even elements of the other kind, flint. Such lenses may be regarded as combinations of achromatic cemented doublets, and formulæ are found for the aberration coefficients of such lenses in terms of those of a standard doublet when the geometrical conditions for the absence of air-gaps between the components are satisfied. Generally speaking, the results reached are that the outer surfaces are concerned with coma, and the internal surfaces with spherical aberration. In all cases the determination of a system to satisfy given conditions involves only the solution of a quadratic equation, and an algebraic method thus effects a solution in a fraction of the time involved in a trigonometrical investigation. Chromatic differences of first order aberrations are easily determined.

The application of the method is illustrated by a series of quadruple objectives which satisfy the ordinary conditions for telescope objectives. Diagrams show the variation of the curvatures with the different forms, the magnitude of the second order spherical aberration, and the chromatic differences of first order aberrations.

DISCUSSION.

Prof. J. W. Nicholson congratulated the author on the increased simplicity which he had brought to some of these important problems. He thought a good case had been made out for the further numerical investigation of this type of objective, and hoped the author would let the Society have the results of such an investigation soon.

Prof. A. E. Conrady (in a communication which was read by the Secretary) said that the Paper could not be regarded as of much practical value. No practical optician would think of constructing a telescope objective of four or more cemented components; even a triple objective is avoided whenever possible, as the technical difficulties are very serious when there are several cemented faces in a lens of considerable size; and he could not conceive of the necessity of such complication ever arising in small lei ses for telescopes.

Mr. T. Smith said a very great deal of work would be involved in a systematic exploration of the properties of multiple lenses over the possible range of optical glasses, and the publication of such an extension could not be expected in a short time. He did not quite understand Prof. Conrady's point of view, which would be most deplorable if it were

generally adopted by the optical industry; for it could only lead to utter stagnation. Obviously there were always difficulties in making a complex rather than a simple instrument; but difficulties must not be allowed to retard progress, and no investigation should be starved out because of them. Unless investigations are made it is impossible to say whether the advantages they may show are sufficient to compensate the attendant disadvantages. As a matter of fact, the question of employing quadruple objectives had actually arisen as a practical proposition in an instrument designed by a practical optician. In the particular case the optical conditions turned out to be unfavourable to the use of such a lens. Had it been otherwise there is no doubt it would have been employed, and any technical difficulties—which were not regarded as a deciding factor—would without doubt have been overcome. In Germany certainly, and in this country also, he believed, quadruple and even quintuple cemented objectives had been made.

An Exhibition of the Uses of Certain Methods of Classification in Optics was given by Mr. T. H. BLAKESLEY, *M.A., at the Meeting on November 23rd,* 1917.

THIS consisted of an account of the additions which, in the course of the intervening years, he had been enabled to make in the general diagram of optical properties, first communicated by him to the Physical Society in the year 1903 ("Proceedings," Vol. XVIII., p. 591). The plan pursued is to take as variables the relations which the radii of face curvature bear to the thickness between the faces along the axis. By this means the shape of the lens is given by the two rectangular co-ordinates alone, and any possible property dependent upon a function of these co-ordinates will be represented by a line upon the diagram. When two such loci intersect, the lens corresponding to the points of intersection possesses both the properties corresponding to the lines. A point much dwelt upon by the author was the very large number of straight-line loci corresponding to properties of value in a lens, and of these very many are parallel, and, cutting the axes at 45 deg., may be most simply defined by the value of the intercept of the axes.

It was pointed out that, in general, a lens may have its radii of face curvature both multiplied by the same factor without changing in sign or value the focal length. One of the above-mentioned loci at 45 deg. to the axes represents the only family in which this change cannot be effected, from the fact that the factor in this case is unity. Another of these straight lines belongs to a family in which the two focal lengths corresponding to two assigned indices of refraction are equal ; and closely allied to this is a family for which the focal length is a minimum for an assigned value of index.

In another family of the kind the property is that a lens may be immersed in another medium without having its focal length changed.

In another, if a lens is cut out of a cylinder of glass, the remnants of the cylinder in their original position will be achromatic.

In another, telescopic ; and so for many others. Other straight lines exist which are not parallel to those above mentioned. They often refer to matters connected with the passage at minimum deviation through a lens, and sometimes to what are called self-conjugate points.

The detection of lens properties which are independent of one of the face curvatures was explained, and some few cases pointed out—*e.g.*, when a lens has one of its radii of face curvature equal to the thickness of the lens at the axis, it matters not what curvature is given to the other face, the point of magnification equal to the index will be coincident with its own conjugate point—*i.e.*, for the point of magnification equal to the inverse of the index for the other side of the lens; and this whichever way the light is passed through the lens.

There are two lines upon the diagram, both straight lines, which refer to the silvering of the second surfaces of lenses, so as to produce plane virtual mirrors; one performs this by sending the centre of the virtual mirror to infinity, the other by sending the surface of the virtual mirror to infinity. In the latter case, which alone calls for special remark, light, though entering the system at an angle, returns upon the same path, always producing an inverted image of -1 magnification, crossing the object at the virtual centre.

DISCUSSION.

Mr. T. Smith suggested that the author might add a number of curves to his diagram showing the aberration properties of lenses. There were a number of other geometrical loci that might also be added. It usually happened that the lenses required in actual instruments had too long radii in comparison with the thickness to be included in the region covered by the author's diagram, and it was usually better to calculate each lens by known methods than to extract them from a diagram.

Mr. S. D. Chalmers said he had on occasion found diagrams somewhat similar to Mr. Blakesley's, but in which the inverse of the radii of curvatures were employed, to be of considerable service in certain problems.

Mr. Blakesley did not think Mr. Chalmers' system would lead to so many straight-line loci as his own. He thought straight-line loci had some advantage if they could be obtained.

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